Mid Term Review

LIMIT

$\lim_{x \to a} f(x)$

1. $\lim_{x \to a^+} f(x) \stackrel{?}{=} \lim_{x \to a^-} f(x)$
2. Squeeze Theorem: Find $g(x)$ and $h(x)$ such that
    1. $h(x) \leq f(x) \leq g(x)$
    2. $\lim_{x \to a} h(x) = \lim_{x \to a} g(x)$
3. L' Hopital's Rule:

    Form: $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 0^0, 1^\infty, \infty^0$

    Notice: $\lim_{x \to a} \frac{f(x)}{g(x)}$: check form and f and g are diff

    Ex. $\lim_{x \to 0} |x|^x$

    Sol. For $x > 0, |x|^x = x^x$

$$\begin{aligned} \lim_{x \to 0^+} |x|^x = \lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{x \ln x} &= e^{\lim_{x \to 0^+} x \ln x} \\ &= \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-x^{-2}} \\ &= \lim_{x \to 0^+} -x = 0 \end{aligned}$$

So, $\lim_{x \to 0^+} |x|^x = e^0 = 1$

For $x < 0, |x|^x = (-x)^x$

$$\lim_{x \to 0^-} |x|^x = \lim_{x \to 0^-} (-x)^x = \lim_{x \to 0^-} e^{x \ln -x} = e^{\lim_{x \to 0^-} x \ln -x}$$

$$= \lim_{x \to 0^-} x \ln -x = \lim_{x \to 0^-} \frac{\ln -x}{\frac{1}{x}} = \lim_{x \to 0^-} \frac{\frac{1}{x}}{-x^{-2}}$$

$$= \lim_{x \to 0^-} -x = 0$$

So, $\lim_{x \to 0^-} |x|^x = e^0 = 1$

Therefore, $\lim_{x \to 0} |x|^x = 1$

4. $\lim_{x \to 0} \frac{\sin x}{x} = 1, \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$
5. Continuity

   If f is continuous at $x = a$

   1. $f(a)$ is defined
   2. $\lim_{x \to 0} f(x)$ exists
   3. $\lim_{x \to 0} f(x) = f(a)$
6. Asymptotes
   1. Vertical

      $f(c)$ is not defined

      check: $\lim_{x \to c^+} f(x) = \pm\infty$ or $\lim_{x \to c^-} f(x) = \pm\infty$
   2. Horizontal

      $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$
   3. Slant

      $$\lim_{x \to \pm\infty} \frac{f(x)}{x} = m \neq 0; \qquad \lim_{x \to \pm\infty} [f(x) - mx] = b$$

      $\Rightarrow y = mx + b$ is a slant asymptote

DIFFERENTIATION

1. f is differentiable at $x = a$

if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ or $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

2. Product rule: $(f \cdot g)' = f'g + fg'$

   Quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

3. Chain rule

   $F(x) = f(g(x))$

   $F'(x) = f'(g(x)) \cdot g'(x)$

4. Implicit differentiation

5. Differentiation of inverse function

   f is differentiable, $g = f^{-1}$. Find $g'(a)$

   1. Find b such that $f(b) = a \to g(a) = b$
   2. $g'(a) = \frac{1}{f'(b)}$
   3. $g'(a) = \frac{1}{f'(g(a))}$

6. Logarithmic differentiation

   Ex. $f(x) = \frac{1}{3}(x-1)^{\frac{1}{3}}(x+1)^{-\frac{2}{3}}(3x+5)$; $f'(x) = ?$

   Sol. Set $y = f(x)$

   $$\ln|y| = \ln|f(x)|$$
   $$(\ln|y|)' = \left(\frac{1}{3}\ln|x-1| - \frac{4}{3}\ln|x+1| + \ln|3x+5|\right)'$$
   $$\frac{y'}{y} = \frac{1}{3} \cdot \frac{1}{x-1} - \frac{4}{3} \cdot \frac{1}{x+1} + \frac{3}{3x+5}$$
   $$y' = \frac{16}{9}(x-1)^{-\frac{2}{3}}(x+1)^{-\frac{7}{3}}$$

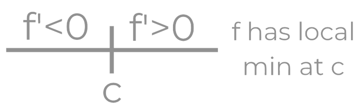
7. Linear approximation of $f(x)$ at $x = a$

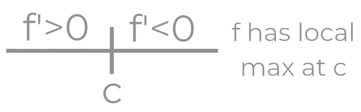
   $L(x) = f(a) + f'(a) + f'(a)(x-a)$

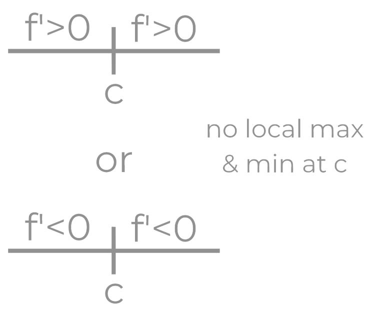
   $f(c) \approx f(a) + f'(a)(c-a)$

8. Related change problem

9. Optimal problem (maximum value, minimum value)

1. c is a critical point of $f(x)$ if
    1. $f(c)$ is defined
    2. $f'(c) = 0$ or $f'(c)$ does not exist
2. First derivative test
    1. 

    2. 

    3. 

3. 2^{nd} derivative test
    1. $f'(c) = 0, f''(c) < 0 : f$ has a local max at c
    2. $f'(c) = 0, f''(c) > 0 : f$ has a local min at c

4. Mean Value Theorem

   f is continuous on $[a, b]$ and is differentiable on (a, b), the there is a $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$

5. Increasing/decreasing

   1. $f'(x) > 0$ on $I \to f$ is increasing on I
   2. $f'(x) < 0$ on $I \to f$ is decreasing on I
   3. If $f'(x) < 0$ or $f'(x) > 0$, then the function is one-to-one and has the inverse function

6. Concavity

   1. $f''(x) > 0$ on I, the graph of f is concave upward on I.
   2. $f''(x) < 0$ on I, the graph of f is concave downward on I.

7. Inflection point