CONTINUITY

//中間值定理

$\lim_{x\to a} f(x) = L \overset{?}{=} L = f(a)$

Def. f is continuous at a number a if $\underbrace{\lim_{x\to a} f(x)}_{\text{exist}} \underset{?}{=} \underbrace{f(a)}_{\text{is defined}}$

To check $f(x)$ is continuous at $x=a$:

1. $f(a)$ is defined

2. $\lim_{x\to a} f(x)$ exists

3. $\lim_{x\to a} f(x) = f(a)$

Obviously, the polynomial and the rational function is continuous at a where a is in its domain.

Ex. Where are discontinuous?

(a) $f(x) = \dfrac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \dfrac{x^2-x-2}{x-2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Sol. (a) Since $f(2)$ is not defined, $f(x)$ is discontinuous at $x=2$.

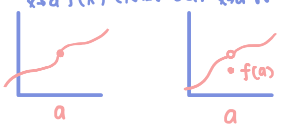
(b) $\lim_{x\to 2} \dfrac{x^2-x-2}{x-2} = \lim_{x\to 2} \dfrac{(x-2)(x+1)}{x-2} = \lim_{x\to 2} (x+1) = 3 \neq 1 = f(2)$

So $f(x)$ is discontinuous at $x=2$.

TYPES of DISCONTINUITY

1. Removable discontinuity

$\lim_{x\to a} f(x)$ exist but $\lim_{x\to a} f(x) \neq f(a)$



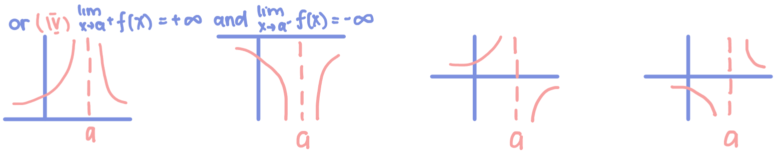
If we redefine $g(x) = \begin{cases} f(x), & x \neq a \\ \lim_{x\to a} f(x), & x = a \end{cases}$

$g(x)$ is continuous at $x=a$.

2. Infinite discontinuity

$\lim_{x\to a} f(x) = +\infty$ (I) or $-\infty$ (II) or (III) $\lim_{x\to a^+} f(x) = -\infty$ and $\lim_{x\to a^-} f(x) = +\infty$

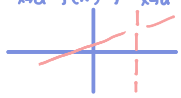
or (IV) $\lim_{x\to a^+} f(x) = +\infty$ and $\lim_{x\to a^-} f(x) = -\infty$



(I) (II) (III) (IV)

3. Jump discontinuity

$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$ but one of $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist.



CONTINUOUS

Def. A function is continuous from the right / left at a, if $\lim_{x\to a^+} f(x) = f(a)$ / $\lim_{x\to a^-} f(x) = f(a)$

Interval: $\underbrace{(a,b)}_{\text{open interval}}$, $[a,b)$, $(a,b]$, $\underbrace{[a,b]}_{\text{close interval}}$

Def. (a) f is continuous on (a,b) if $\lim_{x\to c} f(x) = f(c)$ for all $a<c<b$

(b) f is continuous on $[a,b)$ / $(a,b]$ iff f is continuous on (a,b) and $\lim_{x\to a^+} f(x) = f(a)$ or $\lim_{x\to b^-} f(x) = f(b)$

Thm. The functions f and g are continuous at a, c (constant). Then $f\pm g$, $f\cdot g$, $c\cdot f$, $\frac{f}{g}$ if $g(a)\neq 0$ are continuous at a.

Proof. For $f\cdot g$, $\lim_{x\to a} (f\cdot g)(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = f(a)\cdot g(a)$

Thm. If f is continuous at b and $\lim_{x\to a} g(x) = b = g(a)$, then $\lim_{x\to a} \underbrace{f(g(x))}_{\text{composite function}} = f(\lim_{x\to a} g(x)) = f(b)$

Def. (c) f is continuous on $[a,b]$ if f is continuous on (a,b) $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$

Thm. If f is continuous at $g(a)$ and g is continuous at a, then the composite function $(f\circ g)$ is continuous at a. $\lim_{x\to a} f(g(x)) = g(g(a))$ // $(f\circ g)(x) = f(g(x))$

INTERMEDIATE VALUE

Suppose f is continuous on $[a,b]$ and $f(a)\neq f(b)$. Let $\overset{=0}{N}$ be any number between $\overset{>0}{f(a)}$ and $\overset{<0}{f(b)}$.

Then there is a number c between a and b such that $f(c) = N$ $=0$

Corollary: If f is continuous on $[a,b]$ with $f(a)\cdot f(b) < 0$, then there is a root c between a and b of the equation $f(x)=0$.

Ex. Show there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2

Proof. Set $f(x) = 4x^3 - 6x^2 + 3x - 2$ since $f(x)$ is a polynomial $f(x)$ is continuous on $[1,2]$. $f(1) = -1$.

$f(2) = 12$. Since $f(1)\cdot f(2) < 0$, there is a root c between 1 and 2.

Estimate c:

$f(1)\cdot f(2) < 0 \rightarrow c = 1.\dots$

$f(1.2)\cdot f(1.3) < 0 \rightarrow c = 1.2\dots$

$f(1.22)\cdot f(1.23) < 0 \rightarrow c = 1.22\dots$

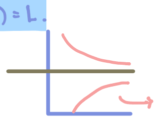
AT INFINITY

// horizontal asymptotes

$\lim_{x\to\infty} f(x) = ?$ and $\lim_{x\to-\infty} f(x) = ?$

Def. Let f be a function defined on some interval $[a,\infty)$. Then $\lim_{x\to\infty} f(x) = L$ $(<\infty)$ / $\lim_{x\to-\infty} f(x) = L$ means for every $\varepsilon > 0$ there is a number N (n) such that if $x > N$ / $x < N$ then $|f(x) - L| < \varepsilon$.

Def. The line $y = L$ is a horizontal asymptote of the curve $y = f(x)$ if either $\lim_{x\to-\infty} f(x) = L$ or $\lim_{x\to\infty} f(x) = L$.



To find the horizontal asymptote of the curve $y=f(x)$, we compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Thm. $r>0$ is a rational number. $\lim_{x\to\infty}\frac{1}{x^r}=0$. Moreover, if x^r is defined for all x, then $\lim_{x\to-\infty}\frac{1}{x^r}=0$.

Def. Let f be a function defined on some interval $[a,\infty)$. Then $\lim_{x\to\infty} f(x)=\infty$ / $-\infty$ means

for every $M>0$ / $m<0$ there is a corresponding N such that if $x>N$ then $f(x)>M$ / $f(x)<m$

$\lim_{x\to\infty} f(x)$

Consider $f(x)$ is a rational function $f(x)=\dfrac{a_nx^n+\cdots+a_1x+a_0}{b_mx^m+\cdots+b_1x+b_0}$

$$\lim_{x\to\infty} f(x)=\begin{cases}0, & n<m\\ \dfrac{a_n}{b_m}, & n=m\\ \lim_{x\to\infty}\dfrac{a_n}{b_m}x^{n-m}, & n>m\end{cases}$$

$$\lim_{x\to-\infty} f(x)=\begin{cases}0, & n<m\\ \dfrac{a_n}{b_m}, & n=m\\ \lim_{x\to-\infty}\dfrac{a_n}{b_m}x^{n-m}, & n>m\end{cases}$$

Ex. Find the vertical and horizontal asymptotes of the graph of the function $f(x)=\dfrac{\sqrt{2x^2+1}}{3x-5}$

Sol. It is easy to see that $f(\frac{5}{3})$ is not defined. Since $\lim_{x\to\frac{5}{3}^+}\dfrac{\sqrt{2x^2+1}}{3x-5}=+\infty$, we have $x=\frac{5}{3}$ is a vertical asymptote. Now, we compute $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

$\lim_{x\to\infty} f(x)$: For $x>0$, $\dfrac{\sqrt{2x^2+1}}{3x-5}\to\dfrac{\sqrt{2}}{3}$

$\lim_{x\to-\infty} f(x)$: For $x<0$, $\dfrac{\sqrt{2x^2+1}}{3x-5}\to-\dfrac{\sqrt{2}}{3}$

Therefore, $y=\pm\frac{\sqrt{2}}{3}$ are the horizontal asymptotes.

Ex. $\lim_{x\to2^+}\tan^{-1}\left(\frac{1}{x-2}\right)$

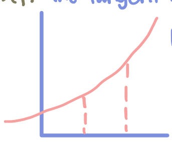
Sol. Set $t=\frac{1}{x-2}$. Then we have $t\to\infty$

$$\lim_{x\to2^+}\tan^{-1}\left(\frac{1}{x-2}\right)=\lim_{t\to\infty}\tan^{-1}t=\frac{\pi}{2}$$

DERIVATIVES &

導數 RATES of CHANGE

Def. The tangent line to the curve $y=f(x)$ at the point $P(a,f(a))$ is the line through P with the slope.



Moreover, the equation of the tangent line is $y-f(a)=m(x-a)$. Set $x=a+h$ as $x\to a$, $h\to0$ so $m=\lim_{h\to0}\dfrac{f(x+h)-f(a)}{h}$

VELOCITY

$f(t)$: position function at time t

Then the velocity $v(t)$ at $t=a$ is $v(a)=\lim_{h\to0}\dfrac{f(a+h)-f(a)}{h}$

Def. The derivative of a function f at a number a, denote it by $f'(a)$ is $f'(a)=\lim_{x\to a}\dfrac{f(x)-f(a)}{x-a}=\lim_{h\to0}\dfrac{f(a+h)-f(a)}{h}$

if the limit exists.

RATE of CHANGE

// $y=f(x)$

If x change from x_1 to x_2, then the change in x is $\Delta x=x_2-x_1$ and the corresponding change in y is $\Delta y=f(x_2)-f(x_1)$. Average rate of change of y with respect to x is $\frac{\Delta y}{\Delta x}$. Instantaneous rate of change $\lim_{\Delta x\to0}\frac{\Delta y}{\Delta x}=\lim_{x_2\to x_1}\frac{f(x_2)-f(x_1)}{x_2-x_1}$. $f'(a)$ = the instantaneous rate of change of $y=f(x)$ with respect to x when $x=a$.

APPLICATION

$f(t)$: position function

$f'(a)$: velocity at time $t=a$

$c(x)$: cost function

$c'(n)$: marginal cost

N : number of bacteria

$N'(a)$: rate of change the number of bacteria

P : population density

$P' \propto P$ → propotional

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Def. The derivative of f is defined by $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

other notation

Leibniz : $y=f(x)$

$f'(x) = y' = \frac{d}{dx} y = \frac{d}{dx} f(x) = Df(x) = D_x f(x)$

$D, \frac{d}{dx}$ = Differentiation operator

Ex. Where is the function $f(x)=|x|$ differentiable?

Sol. For $x>0$, we can choose $|h|$ is small enough such that $x+h>0$

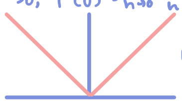
$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = 1$

For $x<0$, we can choose $|h|$ is small enough such that $x+h<0$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{|x+h|-|x|}{h} = \lim_{h\to 0} \frac{-(x+h)-(-x)}{h} = -1$

For $x=0$, $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$ since $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$ and $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$.

So, $f'(0) = \lim_{h\to 0} \frac{|h|}{h}$ does not exist. Therefore, $f(x)$ is differentiable for $x \neq 0$.



$y=|x|$

Thm. If f is differentiable at a, then f is continuous at a. f is continuous $\not\Rightarrow$ f is differentiable.

Ex. $f(x)=|x|$

Proof. To prove f is continuous at a, we want to show $\lim_{x\to a} f(x) = f(a)$ since f is differentiable at a. we have $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ exists.

$\lim_{x\to a} [f(x)-f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot (x-a)$

$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a} (x-a)$

$= f'(a) \cdot 0 = 0$ exists

$\lim_{x\to a} f(x) = \lim_{x\to a} [f(x)-f(a)+f(a)]$

$= \lim_{x\to a} [f(x)-f(a)] + \lim_{x\to a} f(a)$

$= 0 + f(a) = f(a)$

$\therefore$ f is continuous at a.

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$\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

Proof. $\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1-\cos x}{x} \cdot \frac{1+\cos x}{1+\cos x}$

$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \lim_{x\to 0} \frac{\sin^2 x}{x^2} \cdot \frac{x}{1+\cos x} = \frac{0}{1+1} = 0$