LIMIT

·L·I·M·I·T·

$x(t)$: position function

t_0 : given

The average velocity from t_0 to $t_0+\Delta t$ is $\frac{x(t_0+\Delta t)-x(t_0)}{\Delta t}$ if the $\underbrace{\text{average velocity}}_{\text{function}}$ close to a $\underbrace{\text{value}}_{\text{limit}}$ when $\underbrace{\Delta t}_{\text{variable}}$ close to 0, we call the value is the instantaneous velocity at time t_0.

Def. $f(x)$ is defined for x near a. If the function $f(x)$ close to L when x close to a, then we write $\lim_{x\to a} f(x)=L$ or $f(x)\to L$ as $x \xrightarrow{\text{close to}} a$.

$$\boxed{\lim_{x\to a} f(x)=L}$$

- guess L
- prove $\lim_{x\to a} f(x)=L$
- $L=f(a)$?

1. $\lim_{x\to a} f(x)=L \overset{?}{=} L=f(a)$ NO

   Proof :

   $f(x)=\frac{x-1}{x^2-1}$

   $\lim_{x\to 1}\frac{x-1}{x^2-1}=\frac{1}{2}\neq f(1)$

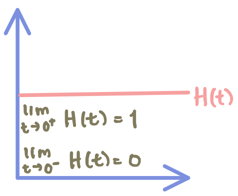
   $f(1)=\frac{1-1}{1^2-1}$ is NOT defined

2. $F(a)=L \overset{?}{=} \lim_{x\to a} f(x)=L$ NO

   Proof :

   $H(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}$

   $H(0)=1$

   When t approach 0 from right / left, $H(t)$ approach 1 / 0.

   $\therefore \lim_{t\to 0} H(t)$ does NOT exist.

Def. //one-sided limit If $f(x)$ close to L when x close to a from right / left

we write $\lim_{x\to a^+} f(x)=L$ / $\lim_{x\to a^-} f(x)=L$.

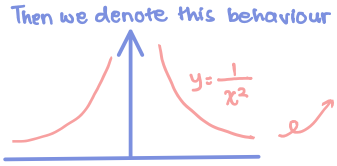
$\lim_{x\to a} f(x)=L$

$\Leftrightarrow \left.\begin{array}{l}\lim_{x\to a^+} f(x)=L\\ \lim_{x\to a^-} f(x)=L\end{array}\right\}$ if and only if

Ex. $\lim_{x\to 0}\frac{1}{x^2}$?

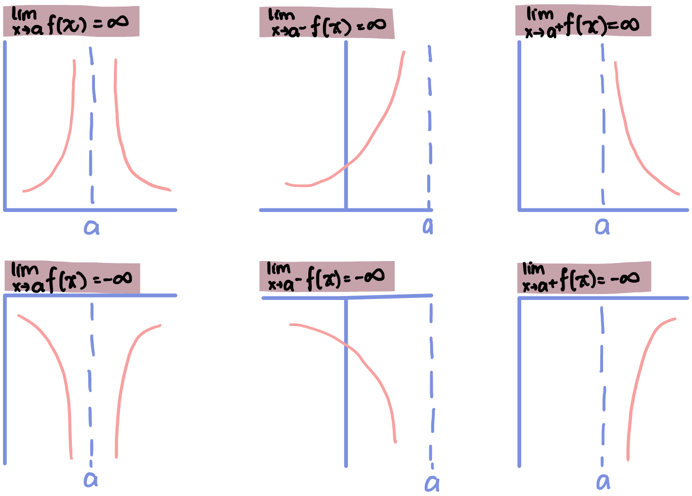
Sol. When x close to zero, we observe $\frac{1}{x^2}$ is increasing and we bounded.

Then we denote this behaviour $\lim_{x\to 0}\frac{1}{x^2}=\infty$.



$x=0$ is a vertical asymptote of $y=\frac{1}{x^2}$

Def. $\lim_{x\to 0} f(x)=\infty$ / $-\infty$ means $f(x)$ can be made arbitrarily large / negative large by taking x sufficiently close to a.



Def. The vertical line $x=a$ is called the asymptote of the curve $y=f(x)$ if one of the following holds: $\lim_{x\to a^+} f(x) = \pm\infty$ or $\lim_{x\to a^-} f(x) = \pm\infty$.

Ex. Find the vertical asymptotes of $f(x) = \tan x$

Sol. $\tan x = \frac{\sin x}{\cos x}$ we know $\cos x = 0$ for $x = \frac{\pi}{2} + n\pi$, $\cos x > 0$. Thus,

$$\lim_{x\to(\frac{\pi}{2})^-} \tan x = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sin x}{\cos x} = +\infty$$

>0 $\to 0$

$x = \frac{\pi}{2}$ is a vertical asymptote of $f(x) = \tan x$.

n is even: The proof is the same with $x = \frac{\pi}{2}$

n is odd: When $x < \frac{\pi}{2} + n\pi$ and close to $\frac{\pi}{2} + n\pi$, $\cos x < 0$, $\sin x < 0$.

$$\lim_{x\to(\frac{\pi}{2}+n\pi)^-} \tan x = \lim_{x\to(\frac{\pi}{2}+n\pi)^-} \frac{\sin x}{\cos x} = +\infty$$

$x = \frac{\pi}{2} + n\pi$ where n is an integer are all the vertical asymptote of $f(x) = \tan x$.

COMPUTING USING LIMIT LAWS

Suppose $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

2. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

3. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$ // c is a constant

4. $\lim_{x\to a} c f(x) = c \lim_{x\to a} f(x)$

5. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, if n is even, $\lim_{x\to a} f(x) \geq 0$.

Prop. If f is a polynomial or a rational function, and a is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$ is defined.

1. If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where n is a positive integer, we call it the polynomial.

2. If $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomial, we call it the rational function.

Ex. (a) $\lim_{x\to 5}(2x^2 - 3x + 4) = 2\cdot 25 - 3\cdot 5 + 4 = 39$

(b) $\lim_{x\to -2}\frac{x^3+2x^2-1}{5-3x} = \frac{-8+8-1}{5+6} = \frac{-1}{11}$

$p(x), q(x)$: polynomials

$\lim_{x\to a}\frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$, if $q(a) \neq 0$

If $q(a) = 0$,

1. $p(a) \neq 0$, $\lim_{x\to a}\frac{p(x)}{q(x)} = +\infty$ or $-\infty$, does not exist.

Ex: $p=1$, $q=x^2$; $p=1$, $q=-x^2$; $p=1$, $q=x$

2. $p(a) = 0$

Prop. If $f(x) = g(x)$ when $x \neq a$, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$ or $\lim_{x\to a}\frac{p(x)}{q(x)} = \lim_{x\to a}\frac{p_1(x)}{q_1(x)}$ where p_1 and q_1 satisfy $p(x) = (x-a)p_1(x)$ and $q(x) = (x-a)q_1(x)$.

Ex. $\lim_{x\to 1}\frac{x^2-1}{x-1}$

Sol. When $x=1$, $x^2-1=0$, and $x-1=0$

$x^2-1 = (x-1)(x+1)$

$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)}{x-1} = \lim_{x\to 1}(x+1) = 2$

Ex. $\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}$ // $a^2-b^2 = (a-b)(a+b)$

Sol. When $t=0$, $\sqrt{t^2+9}-3 = 0$ and $t^2=0$.

$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\times\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3} = \lim_{t\to 0}\frac{\cancel{t^2}\cancel{+9-9}}{\cancel{t^2}(\sqrt{t^2+9}+3)} = \frac{1}{6}$

Ex. $\lim_{x\to 0}\frac{|x|}{x}$ does not exist

Proof. $\lim_{x\to 0}\frac{|x|}{x}$ for $x>0$, $|x| = x$ so $\lim_{x\to 0^+}\frac{|x|}{x} = 1$

$\lim_{x\to 0}\frac{|x|}{x}$ for $x<0$, $|x| = -x$ so $\lim_{x\to 0^-}\frac{|x|}{x} = -1$

Since $\lim_{x\to 0^+}\frac{|x|}{x} = 1 \neq -1 = \lim_{x\to 0^-}\frac{|x|}{x}$, $\lim_{x\to 0}\frac{|x|}{x}$ does not exist.

// $[\![x]\!]$ or $[x]$ or $\lfloor x\rfloor$ greatest integer function ; the largest integer that is less than x.

// $\lceil x\rceil$ least integer function ; the smallest integer that is more than x.

// Ex. (1) $\lceil -2.5\rceil = -2$

(2) $\lfloor \pi \rfloor = 3$

1. If $f(x) \le g(x)$ when x is near a and both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$.

   Ex. Assume $f(x) \le 0$ when x is near a and $\lim_{x\to a} f(x)$ exists. By <1>, $\lim_{x\to a} f(x) \le 0$.

2. Squeeze Theorem

   If $f(x) \le \underset{\text{find}}{\boxed{g(x)}} \le h(x)$ when x is near a and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$

then $\lim_{x\to a} g(x) = L$.

Ex. Show $\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

Proof. First we know that $-1 \le \sin\frac{1}{x} \le 1$. So we have $\overset{f(x)}{\boxed{-x^2}} \le x^2 \sin\frac{1}{x} \le \overset{h(x)}{\boxed{x^2}}$.

Also, we have $\lim_{x\to 0}(-x^2) = 0 = \lim_{x\to 0} x^2$. By Squeeze Theorem,

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$.

P·R·E·C·I·S·E·D·E·F·I·N·I·T·I·O·N

If $f(x)$ is arbitrarily close to L by restricting x to be sufficiently close to a, $\lim_{x\to a} f(x) = L$. For every $\varepsilon > 0$, $\exists\, \delta > 0$ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$.

To show $\lim_{x\to a} f(x) = L$, given any $\varepsilon > 0$, our goal is to find $\delta > 0$ such that the statement if $0 < |x-a| < \delta$ then $|f(x) - L| < \varepsilon$ holds.

Known: a, $f(x)$, L, ε

Unknown: δ

1. Observe the relation between $|f(x) - L|$ and $|x-a|$.

2. Use the relation found in <1> to guess δ.

3. Show the δ works.

Ex. Show $\lim_{x\to 3} x^2 = 9$

Proof. $|x^2 - 9| = |x-3||x+3|$

For $|x-3| < 1$, we have $2 < x < 4$ and $5 < |x+3| < 7$ so $|x^2-9| < 7|x-3|$. Since we want $|x^2 - 9| < \varepsilon$, we ask for $|x-3| < \frac{\varepsilon}{7}$. Now we take $\delta = \min\{1, \frac{\varepsilon}{7}\}$.

If $0 < |x-3| < \delta$, then $|x^2-9| = |x-3||x+3| < 7|x-3|$

since $|x-3| < \delta < 1 \to |x+3| < 7$

$< 7\delta \le 7 \cdot \frac{\varepsilon}{7} = \varepsilon$

$\delta = \min\{1, \frac{\varepsilon}{7}\}$

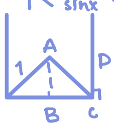
Therefore, $\lim_{x\to 3} x^2 = 9$.

$\lim_{x\to a} f(x) = \infty \,/\, -\infty$ means that for every $N > 0 \,/\, N < 0$ there is a $\delta > 0$ such that if $0 < |x-a| < \delta$, then $f(x) > N \,/\, f(x) < N$.

Ex. Show $\lim_{x\to 0} \frac{\sin x}{x} = 1$

Proof. For $x > 0$, since $\overline{AB} < \overset{\frown}{AC} < \overline{DC}$ we have $\sin x < x < \tan x$. It shows us

$1 < \frac{x}{\sin x} < \frac{1}{\cos x} \to 1 > \frac{\sin x}{x} > \cos x$



Since $\lim_{x\to 0^+} 1 = 1$, $\lim_{x\to 0^+} \cos x = 1$ by Squeeze Theorem, we obtain $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$. It remains to show $\lim_{x\to 0^-} \frac{\sin x}{x} = 1$. Set $y = -x$. As $x \to 0^-$, we have $y \to 0^+$.

$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{y\to 0^+} \frac{\overbrace{\sin(-y)}^{-\sin y}}{-y} = \lim_{y\to 0^+} \frac{\sin y}{y} = 1$

From the results shown in the above, we have $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Corollary: $\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

Proof. $\lim_{x\to 0} \frac{\sin ax}{bx} = \lim_{x\to 0} \frac{a}{b} \cdot \frac{\sin ax}{ax} = \frac{a}{b} \cdot \lim_{x\to 0} \frac{\sin ax}{ax} = \frac{a}{b}$