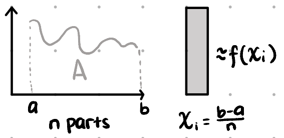
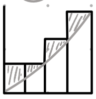
INTEGRAL

AREA & DISTANCE



$\text{Area} = \frac{b-a}{n} f(x_i)$

Q Find the area lies under the curve $y=f(x)$ from a to b



We can observe that the approximating rectangle approaches the region as n approach infinity.

To compute the area A of the region S which is under the curve $y=f(x)$, we first divide the interval $[a,b]$ into n-subinterval. That is we choose

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_n = b \quad \text{where } x_i = a + i\frac{b-a}{n},\ i = 0, \dots, n.$$

Then the area $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\underbrace{\Delta x_i}_{\frac{b-a}{n}}$ $\quad x_{i-1} \le x_i^* \le x_i$: sample point

If $f(x_i^*)$ is the maximum / minimum value of f on $[x_{i-1}, x_i^*]$, the sum $\sum_{i=1}^{n} f(x_i^*)\Delta x_i$ is called the uppersum / lowersum

$$\text{lowersum} \le A \le \text{uppersum}$$

Ex. $f(x) = e^{-x}$; $x = 0 \sim x = 2$; Find the area A

(a) Using right-end point. Find A as a limit.

(b) Using midpoints four-subinterval $(n=4)$, ten-subinterval $(n=10)$

Sol. (a) $b=2$; $a=0$

$\Delta x_i = \frac{2-0}{n} = \frac{2}{n}$

n-subinterval : $[0, \frac{2}{n}], [\frac{2}{n}, \frac{4}{n}], \dots, [\frac{2n-2}{n}, 2]$

right-endpoint for $[x_{i-1}, x_i]$ is $\frac{2i}{n}$. So, $x_i^* = \frac{2i}{n}$

$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{2}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} e^{-\frac{2i}{n}} \cdot \frac{2}{n} = \lim_{n\to\infty} \frac{2}{n}(e^{-\frac{2}{n}} + e^{-\frac{4}{n}} + \cdots + e^{-2}) = 1 - e^{-2}$

(b) 4-subinterval : $[0, \frac{1}{2}], [\frac{1}{2}, 1], [1, \frac{3}{2}], [\frac{3}{2}, 2]$

use midpoint : we have $x_1^* = \frac{1}{4}$, $x_2^* = \frac{3}{4}$, $x_3^* = \frac{5}{4}$, $x_4^* = \frac{7}{4}$.

$M_4 = \frac{2-0}{4}[f(\frac{1}{4}) + f(\frac{3}{4}) + f(\frac{5}{4}) + f(\frac{7}{4})] = \frac{1}{2}(e^{-\frac{1}{4}} + e^{-\frac{3}{4}} + e^{-\frac{5}{4}} + e^{-\frac{7}{4}})$

10-subinterval : $\Delta x = \frac{2-0}{10} = \frac{1}{5}$

subinterval : $[0, 0.2], [0.2, 0.4], \dots, [1.8, 2]$

midpoint : $0.1, 0.3, \dots, 1.9$

$M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + \cdots + f(1.9)] = \frac{1}{5}(e^{-0.1} + \cdots + e^{-1.9})$

DEFINITE

Def. If f is defined on $[a,b]$, we divide $[a,b]$ into n-subinterval of equal width $\Delta x = \frac{b-a}{n}$. Let $a = x_0 < x_1 < x_2 < \cdots < x_n = b$, $x_i = a + i\Delta x_i$ be the endpoints of these subintervals and let $x_1^*, x_2^*, \dots, x_n^*$ be any sample points in these subinterval $(x_{i-1} \le x_i^* \le x_i)$. The definite integral of f from a to b is $\int_a^b f(x)\,dx = \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x}_{\text{riemann sum}}$ (upper: b, lower: a) if the limit exists. Moreover,

we say that f is integrable on $[a,b]$.

Ex. $f(x) = \begin{cases} 1, & x \text{ is rational number} \\ 0, & x \text{ is irrational number} \end{cases}$

Prove $f(x)$ is not integrable on $[0,1]$

Proof. First, we divide $[0,1]$ into n-subinterval. $\Delta x = \frac{1}{n}$. We take x_i^* is a rational number between $[x_{i-1}, x_i]$ where $x_i = \frac{i}{n}$.

$\lim_{n\to\infty} \sum_{i=1}^{n} \overset{=0}{f(x_i^*)} \cdot \frac{1}{n} = 0$. So, f is not integrable on $[0,1]$.

Method to prove f is integrable on $[a,b]$:

Let $\bar{x}_i$ be the sample point such that $f(\bar{x}_i) = \max_{x\in[x_{i-1}, x_i]} f(x)$

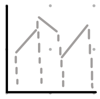
Then we have uppersum $\sum_{i=1}^{n} f(\bar{x}_i) = \Delta x_i$

Let $\underline{x}_i$ be the sample point such that $f(\underline{x}_i) = \min_{x\in[x_{i-1}, x_i]} f(x)$

Then we have lowersum $\sum_{i=1}^{n} f(\underline{x}_i)\Delta x = L_n$; $L_n \le \sum_{i=1}^{n} f(x_i^*)\Delta x \le U_n$

If $\lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$ by Squeeze Theorem, f is integrable on $[a,b]$.

Thm. If f is continuous on $[a,b]$ or f has only a finite jump discontinuities then f is integrable on $[a,b]$. That is, $\int_a^b f(x)\,dx$ exists.



2 jump discontinuity

Thm. If f is integrable on $[a,b]$ then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i$

$$\Delta x = \frac{b-a}{n} \ ; \ x_i = a + i\Delta x$$

Ex. $\int_0^3 (x^3 - 6x)\,dx$

Sol. Since x^3-6x is a polynomial x^3-6x is continuous on $[0,3]$. So, x^3-6x is integrable on $[0,3]$. Therefore, $\Delta x = \frac{3-0}{n} = \frac{3}{n}$. $x_i = 0 + i\frac{3}{n} = \frac{3i}{n}$.

By Thm 4, we have

$$\begin{aligned}
\int_0^3 (x^3-6x)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \cdot \frac{3}{n} \\
&= \lim_{n\to\infty} \frac{3}{n^2}\left(\sum_{i=1}^{n} 27\frac{i^3}{n^2} - 18i\right) \\
&= \lim_{n\to\infty} 81\frac{\sum_{i=1}^{n} i^3}{n^4} - 54\frac{\sum_{i=1}^{n} i}{n^2} \\
&= \lim_{n\to\infty} 81 \cdot \frac{n^2(n+1)^2}{4n^4} - 54 \cdot \frac{n(n+1)}{2n^2} = \frac{81}{4} - 27 = -\frac{27}{4}
\end{aligned}$$

Ex. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2 n^3}{i^6 + n^6}$

Sol. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2 n^3}{i^6+n^6} = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2 n^4}{i^6+n^6} \cdot \overset{\Delta x}{\frac{1}{n}} = \lim_{n\to\infty} \frac{(\frac{i}{n})^2}{1+(\frac{i}{n})^6} \cdot \frac{1}{n}$

Consider $x_i = \frac{i}{n}$ $f(x) = \frac{x^2}{1+x^6}$; $a = x_0 = 0$; $b = x_n = 1$

By definition of Riemann sum, $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2n^3}{i^6+n^6} = \int_0^1 \frac{x^2}{1+x^6}\,dx$

PROPERTIES

- $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b c\,dx = c(b-a)$
- $\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx$
- $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
- If $f(x) \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge 0$
- If $f(x) \ge g(x)$ on $[a,b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$ (set $h = f-g \ge 0$; $\int_a^b h \ge 0$; $\int_a^b f \ge \int_a^b g$)

  $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$
- If $m \le f(x) \le M$, then $\int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx = M(b-a)$

$g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \Rightarrow g(x+h) - g(x) \approx f(x)\cdot h = f(x)$

FUNDAMENTAL

Thm. FCT, part I

If f is continuous on $[a,b]$. Define $g(x) = \int_a^x f(t)\,dt$ $a \le x \le b$. Then $g(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Moreover, $g'(x) = f(x)$.

Proof. We first prove g is differentiable on (a,b). Given $x_0 \in (a,b)$. Then we can find $h>0$ sufficiently small such that $a < x+h < b$. Then f is continuous on $[a,x]$ and $[a, x+h]$.

$$g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

Since f is continuous on $[x, x+h]$, there is u and v in $[x, x+h]$ such that $f(u) = \max_{t\in[x,x+h]} f(t)$ and $f(v) = \min_{t\in[x,x+h]} f(t)$ extreme value theorem

$$f(v) \le f(t) \le f(u) \quad \forall t \in [x, x+h]$$

$$f(v)h \le \int_x^{x+h} f(t)\,dt \le f(u)h$$

Since $h>0$, we have $f(v) \le \frac{g(x+h)-g(x)}{h} \le f(u)$. As $h\to 0$, $u, v \to x$.

Since f is continuous $f(u), f(v) \to f(x)$. By Squeeze Theorem,

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = f(x)$$

Similarly,

$$\lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} = f(x)$$

Therefore. $g'(x) = f(x)$ $a<x<b$. Hence, f is differentiable on (a,b). Also, f is continuous on (a,b). $g(x) = \int_a^x f(t)\,dt$. It remains to show

$$\underbrace{\lim_{x\to a^+} g(x)}_{\lim_{x\to a^+} g(a+h)} = \underbrace{g(a)}_{0}$$

and $\lim_{x\to b^-} g(x) = g(b)$. For $h>0$, since f is continuous on $[a, a+h]$, there is u and v such that $\underbrace{f(v)h}_{\text{min}} \le \int_a^{a+h} f(t)\,dt \le \underbrace{f(u)h}_{\text{max}}$. As $h\to 0^+$,

$$\int_a^{a+h} f(t)\,dt \to 0 = g(a).$$

So, f is right continuous at $x=a$. Similarly, we can show $\lim_{x\to b^-} g(x) = g(b)$.

There fore, f is continuous on $[a,b]$.

Conclusion: f is continuous on $[a,b]$, $g(x) = \int_a^x f(t)\,dt$ is an antiderivative of f.

Moreover, $\frac{d}{dx} g(x) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$. So, $g(x)$ is antiderivative of f.

Ex. $\frac{d}{dx}\left(\int_1^{x^4} \sec t\,dt\right)$

Sol. Let $u = x^4$ chain rule

$\frac{d}{dx}\left(\int_1^{x^4} \sec t\,dt\right) = \frac{d}{dx}\left(\int_1^{u} \sec t\,dt\right) = \frac{d}{du}\left(\int_1^{u} \sec t\,dt\right)\frac{du}{dx} = \sec u \cdot 4x^3 = 4x^3 \sec x^4$

Thm. FCT, part 2

If f is continuous on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$ where F is an antiderivative of f.

Proof. Define $g(x) = \int_a^x f(t)\,dt$. Then $g(x)$ is an antiderivative of f (by FCT part 1). So, we have $F(x) = g(x) + C$. $F(b) - F(a) = g(b) - \overset{0}{g(a)} = \int_a^b f(t)\,dt$.

Ex. $\int_1^3 \frac{1}{x}\,dx$

Sol. $\ln|x|$ is an antiderivative of $\frac{1}{x}$.

$\int_1^3 \frac{1}{x}\,dx = \left[\ln|x|\right]_1^3 = \ln|3| - \ln|1| = \ln 3$

INDEFINITE

Def. $\int f(x)\,dx$ is the antiderivative of f. It is called the indefinite integral. F is antiderivative of f. $\int f(x)\,dx = F(x) + C$

Ex. $\frac{1}{2}x^2$ is an antiderivative of x

Sol. $\int x\,dx = \frac{1}{2}x^2 + C$

Ex. $\int 10x^4 - 2\sec^2 x\,dx$

Sol. $\int 10x^4 - 2\sec^2 x\,dx = 10\int x^4\,dx - 2\int \sec^2 x\,dx = 2x^5 - 2\tan x + C$

Ex. $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt$

Sol. $\int_1^9 \frac{2t^2 + t^2\sqrt{t} - 1}{t^2}\,dt = \int_1^9 (2 + t^{\frac{1}{2}} - t^{-2})\,dt = \left[2t + \frac{2}{3}t^{\frac{3}{2}} + t^{-1}\right]_1^9 = 32\frac{4}{9}$

NET CHANGE

Rate of change of a quantity $F(x)$ is $F'(x)$.

The net change of $F(x)$ from a to b is $F(b) - F(a) = \int_a^b F'(x)\,dx$

SUBSTITUTION

$\int 2x\sqrt{1+x^2}\,dx$ $\int \frac{x^2}{1+x^6}\,dx$

Idea: Set $u = 1+x^2$; $\frac{du}{dx} = 2x \to du = 2x\,dx$

$\int \underbrace{\sqrt{1+x^2}}_{\sqrt{u}}\, \underbrace{2x\,dx}_{du} = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

Goal: $\int F(x)\,dx$

Idea: Find suitable $u = g(x)$ and $f(u)$ such that $F(x) = f(g(x)) \cdot g'(x)$. Then $du = g'(x)\,dx$.

$\int f(\underbrace{g(x)}_{u}) \cdot \underbrace{g'(x)\,dx}_{du} = \int f(u)\,du$

Ex. $\int x^3\cos(x^4+2)\,dx$

Sol. Set $u = x^4+2$; $du = 4x^3\,dx$

$\int x^3\cos(x^4+2)\,dx = \int \cos u \cdot \frac{1}{4}\,du = \frac{1}{4}\sin u + C = \frac{1}{4}\sin(x^4+2) + C$

Ex. $\int e^{kx}\,dx$, $k \ne 0$

Sol. Set $u = kx$; $du = k\,dx$

$\int e^{kx}\,dx = \int e^u \cdot \frac{1}{k}\,du = \frac{1}{k}e^u + C = \frac{1}{k}e^{kx} + C$

Ex. $\int \tan x\,dx$

Sol. $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

Set $u = \cos x$; $du = -\sin x\,dx$

$\int \tan x\,dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \ln|\sec x| + C$

Ex. $\int \sec x\,dx$

Sol. $\int \sec x\,dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x}\,dx$

Set $u = \sec x + \tan x$; $du = \sec x\tan x + \sec^2 x\,dx$

$\int \sec x\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sec x + \tan x| + C$

Ex. $\int \sqrt{1+x^2}\,x^5\,dx$

Sol. Set $u = 1+x^2$; $du = 2x\,dx$

$\int \sqrt{1+x^2}\,x^5\,dx = \int \sqrt{1+x^2}\,x^4 \cdot x\,dx = \int \sqrt{u}\,(u-1)^2 \cdot \frac{du}{2}$

$= \frac{1}{2}\int u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + u^{\frac{1}{2}}\,du = \frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}} - \frac{4}{5}u^{\frac{5}{2}} + \frac{2}{3}u^{\frac{3}{2}}\right) + C$

$= \frac{1}{7}(1+x^2)^{\frac{7}{2}} - \frac{2}{5}(1+x^2)^{\frac{5}{2}} + \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$

$\int_a^b f(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$; $u = g(x)$

Ex. $\int_1^2 \frac{dx}{(3-5x)^2}$

Sol. Set $u = 3-5x$; $du = -5\,dx$

$x = 1 \to u = -2$; $x = 2 \to u = -7$

$\int_1^2 \frac{dx}{(3-5x)^2} = \int_{-2}^{-7} \frac{1}{u^2}\left(-\frac{1}{5}du\right) = -\frac{1}{5}\int_{-2}^{-7} \frac{du}{u^2} = -\frac{1}{5}\left[-\frac{1}{u}\right]_{-2}^{-7} = \frac{1}{14}$

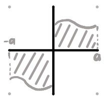
Ex. $\int_0^1 \frac{x^2}{1+x^6}\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2 n^3}{i^6 + n^6}$

Sol. Set $u = x^3$; $du = 3x^2\,dx$

$\int_0^1 \frac{x^2}{1+x^6}\,dx = \int_0^1 \frac{1}{1+u^2} \cdot \frac{du}{3} = \frac{1}{3}\left[\tan^{-1} u\right]_0^1 = \frac{\pi}{12}$

f is continuous on $[-a, a]$

1. If f is even ($f(x) = f(-x)$)
$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$



2. If f is odd ($f(x) = -f(-x)$)
$\int_{-a}^{a} f(x)\,dx = 0$