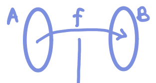
FUNCTIONS

EXPONENTIAL

Def. An exponential function is a function of the form $f(x) = b^x$ where $b>0$.



A: domain 定義域

B: range 值域

↳ relation between A & B

For the exponential function

Domain is $(-\infty, \infty)$

Range is $(0, \infty)$

LAW of EXPONENTS

$a, b > 0$, $x, y \in (-\infty, \infty)$ then

(i) $b^{x+y} = b^x \cdot b^y$ (iii) $(b^x)^y = b^{xy}$

(ii) $b^{x-y} = \frac{b^x}{b^y}$ (iv) $(ab)^x = a^x b^x$

APPLICATIONS

- Population of bacteria
- Half life

Half life of ^{90}Sr 25 years $m(0) = 24mg$

(a) $m(t)$ = mass remains t years ; find $m(t)$

(b) $m(40)$

(c) estimate t year such that $m(t) = 5$

Sol. (a) after 1 period is 25 years, the remain mass is $25 \times \frac{1}{2}$

after n period, the mass $25 \times (\frac{1}{2})^n$, t years is $\frac{t}{25}$

Period. $m(t) = 24 \times (\frac{1}{2})^{\frac{t}{25}}$

(b) $m(40) = 24 \times (\frac{1}{2})^{\frac{40}{25}} = 24 \times (\frac{1}{2})^{\frac{8}{5}}$

(c) $24(\frac{1}{2})^{\frac{t}{5}} = 5 \Rightarrow t = 57$

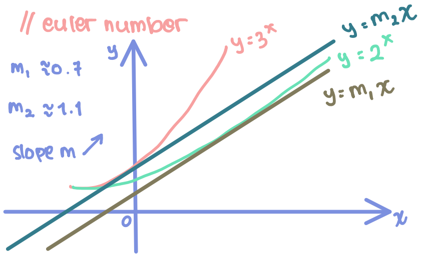
THE NUMBER e

// euler number

$m_1 \approx 0.7$

$m_2 \approx 1.1$

slope m ↗



$y = m_2 x \rightarrow$ tangent to $y = 3^x$

$y = m_1 x \rightarrow$ tangent to $y = 2^x$

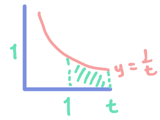
// ∃! ← exist, ↳ unique

Def. ∃! number b with $2 < b < 3$ such that the slope of the tangent line to $y = b^x$ at $(0,1)$ is 1, here we call it $e = 2.71828$

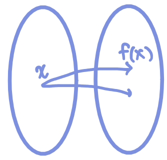
// e^x : natural exponential function

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{t\to 0}(1+t)^{\frac{1}{t}} = \sum_{n=0}^{\infty}\frac{1}{n!}$$

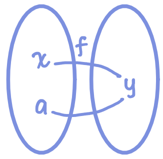
e = the unique number a, such that $\int_1^a \frac{1}{t}\,dt = 1$ // a=e

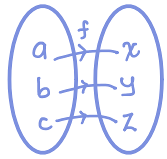


ONE TO ONE FUNCTION



is NOT function





1-1 function

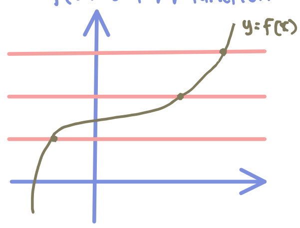
Def. A function f is called a 1-1 function if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ (or if $f(x_1) = f(x_2)$, then $x_1 = x_2$).

Test for 1-1 function

1. Choose $x_1 \neq x_2$, check $f(x_1) \neq f(x_2)$.

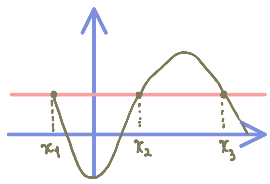
2. Assume $f(x_1) = f(x_2)$, check $x_1 = x_2$.

$\Rightarrow f(x)$ is a 1-1 function



Horizontal line test no more than one intersection.

$\Rightarrow f$ is 1-1.



$f(x_1) = f(x_2) = f(x_3)$

BUT $x_1 \neq x_2 \neq x_3$

$\Rightarrow f$ is NOT 1-1

Ex. $f(x) = x^3$; is it 1-1?

Sol. // use way 1

Choose any $x_1 \neq x_2$ without loss of generality (WLOG), we assume $x_2 > x_1$.

$$f(x_2) - f(x_1) = x_2^3 - x_1^3$$

$$= \underbrace{(x_2 - x_1)}_{>0}\underbrace{(x_2^2 + x_2 x_1 + x_1^2)}_{>0}$$

$f(x_1) \neq f(x_2)$

f is a 1-1 function.

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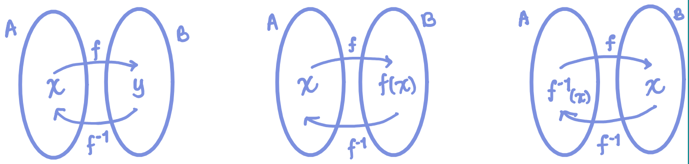
Def. Let $f: A \to B$ be a 1-1 function then its inverse function $f^{-1}: B \to A$ is defined by $f^{-1}: B \to A \Leftrightarrow f(x) = y$ for any y in B.

Ex. Known $f(1) = 10$, $f^{-1}(10) = 1$



$f^{-1}(f(x)) = x$

$x \in A$

$f(f^{-1}(x)) = x$

$x \in B$

Find the inverse function of a 1-1 function $f(x)$:

1. Set $y = f(x)$ solve the equation to obtain $x = g(y)$

   // g is unique

2. Then the inverse function $f^{-1}(x) = g(x)$

Ex. $g(x) = 3 + x + e^x$; find $g^{-1}(4)$?

Sol. $g^{-1}(x) = k$

$g(k) = x$

$g(k) = 4 \Leftrightarrow g^{-1}(4) = k$

$4 = 3 + k + e^k$

$1 = k + e^k$

$k = 0$

$g^{-1}(4) = 0$ //

Ex. Find a formula for the inverse of the function $y = \frac{e^x}{1 + 2e^x}$

Sol. $x = \frac{e^y}{1 + 2e^y} : e^y \rightarrow x = \frac{1}{e^{-y} + 2}$

$e^{-y} + 2 = \frac{1}{x}$

$e^{-y} = \frac{1}{x} - 2$

$e^{-y} = \frac{1 - 2x}{x}$

$-y = \log_e\left(\frac{1-2x}{x}\right)$

$y = -\log_e\left(\frac{1-2x}{x}\right)$

$y = \log_e\left(\frac{x}{1-2x}\right)$ //

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Def. $\log_b x$ is the inverse function of the function $f(x) = b^x$, $b > 0$, $b \neq 1$.

When $b = e$, $\log_e x = \ln x$ is the natural logarithmic function

$\log_b x = \frac{\ln x}{\ln b}$

$\ln e = 1$

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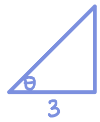
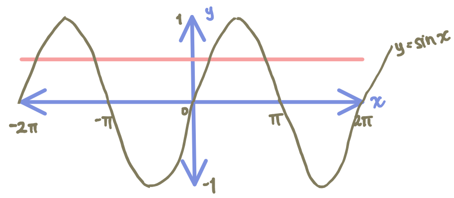
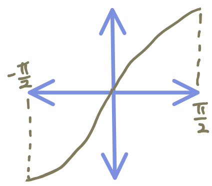


$1 \rightarrow \theta = ?$





$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$-1 \le x \le 1$

Ex. $\sin^{-1}\frac{1}{2} = ?$

Sol. Set $\sin^{-1}\frac{1}{2} = y \Leftrightarrow \sin y = \frac{1}{2}$, between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, we only have $\sin\frac{\pi}{6} = \frac{1}{2}$. So, $y = \frac{\pi}{6}$.

	Domain	Range
$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \sec^{-1} x$	$\lvert x \rvert \ge 1$	$0 \le y < \frac{\pi}{2}$, $\pi \le y < \frac{3\pi}{2}$
$y = \csc^{-1} x$	$\lvert x \rvert \ge 1$	$0 < y \le \frac{\pi}{2}$, $\pi < y \le \frac{3\pi}{2}$

Ex. simplify $\cos(\tan^{-1} x)$

Sol. Set $\tan^{-1} x = y \Rightarrow \tan y = x$

$\underbrace{\cos y}_{>0} = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{1}{\sqrt{1 + x^2}}$

$\sec^2 y = 1 + \tan^2 y$

$\sec y = \pm\sqrt{1 + \tan^2 y}$

$\sec y = \sqrt{1 + \tan y}$

$\therefore \cos(\tan^{-1} x) = \frac{1}{\sqrt{1 + x^2}}$