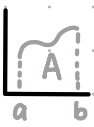
APPLICATION

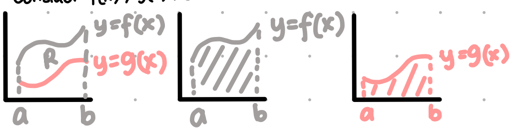
Area 2 curves



$y=f(x)\geq 0$

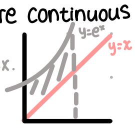
$A=\int_a^b f(x)\,dx$

Consider $f(x)\geq g(x)\geq 0$



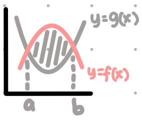
The area R of the region bounded by $y=f(x)$, $y=g(x)$, $x=a$, $x=b$ where f and g are continuous on $[a,b]$ and $f(x)=g(x)$ for all x in $[a,b]$ is $A=\int_a^b (f-g)(x)\,dx$.

Ex.



Sol. $e^x>x$ for $0\leq x\leq 1$

The area is $\int_0^1 e^x-x\,dx = e^x-\frac{1}{2}x^2\Big]_0^1 = e-\frac{3}{2}$



$f(x)=g(x)$ → for finding a & b

Region enclosed by $y=f(x)$ and $y=g(x)$

1. Solve $f(x)=g(x)$

2. The area is $\int_a^b (f-g)(x)\,dx$

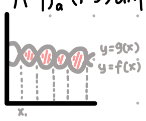
Ex. Find the area of the region enclosed by $y=x^2$ and $y=2x-x^2$

Sol. First, we solve $x^2=2x-x^2$. Then we have $x=0$ and $x=1$. Choose $x=0.5$, we have $0.5^2=0.25<0.75=0.5*2-0.5^2$. So, we have $x^2\leq 2x-x^2$ for all x in $[0,1]$. The area is $\int_0^1 2x-x^2-x^2\,dx = x^2-\frac{2}{3}x^3\Big]_0^1=\frac{1}{3}$

The area of the region enclosed by $y=f(x)$ and $y=g(x)$ ($x=a$ & $x=b$)

1. Solve $f(x)=g(x)$ ($f(x)=g(x)$ for $a\leq x\leq b$)

2. Assume $x_1<x_2<\cdots<x_n$ are the roots of $f(x)=g(x)$. Then, the area is
$A=\left|\int_a^{x_1}(f-g)\,dx\right|+\left|\int_{x_1}^{x_2}(f-g)\,dx\right|+\cdots+\left|\int_{x_n}^{b}(f-g)\,dx\right|$



$A_i=\left|\int_{x_i}^{x_{i+1}}(f-g)(x)\,dx\right|$

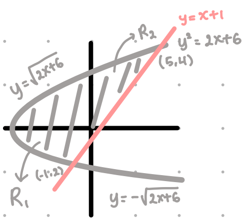
Ex. $y=\sin x$, $y=\cos x$, $x=0$, $x=\frac{\pi}{2}$

Sol. First, we solve $\sin x=\cos x$ for $0\leq x\leq\frac{\pi}{2}$. We have $x=\frac{\pi}{4}$. The area is

$A=\left|\int_0^{\frac{\pi}{4}}(\sin x-\cos x)\,dx\right|+\left|\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}(\sin x-\cos x)\,dx\right|$

$=\left|-\cos x-\sin x\Big]_0^{\frac{\pi}{4}}\right|+\left|-\cos x-\sin x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right|=2(\sqrt{2}-1)$

Ex



$R_1: -3\leq x\leq -1$

$-\sqrt{2x+6}\leq y\leq\sqrt{2x+6}$

$R_2: -1\leq x\leq 5$

$x+1\leq y\leq\sqrt{2x+6}$

Sol. Area $=\int_{-3}^{-1}\sqrt{2x+6}-(-\sqrt{2x+6})\,dx+\int_{-1}^{5}\sqrt{2x+6}-(x-1)\,dx$

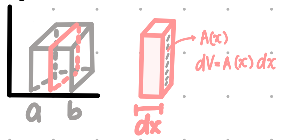
$R: -2\leq y\leq 4$ $\left(\begin{matrix}y+1=\frac{y^2-6}{2}\\ y=-2,4\end{matrix}\right)$

$\frac{y^2-6}{2}\leq x\leq y+1$

Area $=\int_{-2}^{4} y+1-\frac{y^2-6}{2}\,dy=\frac{y^2}{2}-\frac{y^3}{6}+4y\Big]_{-2}^{4}=18$

Volume

1. Cross-section



$V=\int_a^b A(x)\,dx$

$A(x)$: the area of the cross-section

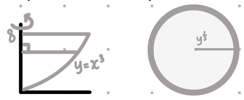
Def. Let S be a solid that lies between $x=a$ and $x=b$. If the cross-section area of S in the plane P_x through x and perpendicular to the x-axis is $A(x)$, where A is a continuous function, then the volume of S is $\int_a^b A(x)\,dx$.

Ex. Show the volume of a ball with the radius r is $\frac{4}{3}\pi r^3$

Sol. $A(x)=\pi(\sqrt{r^2-x^2})^2=\pi(r^2-x^2)$

$V=\int_{-r}^{r}A(x)\,dx=\pi\int_{-r}^{r}r^2-x^2\,dx=\pi\left[r^2x-\frac{1}{3}x^3\right]_{-r}^{r}=\frac{4}{3}\pi r^3$

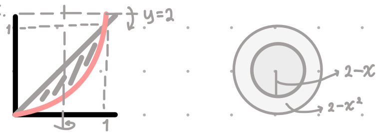
Ex.



Sol. $A(y)=\pi(\sqrt[3]{y})^2=\pi y^{\frac{2}{3}}$

$V=\int_0^8\pi y^{\frac{2}{3}}\,dy=\frac{3}{5}\pi y^{\frac{5}{3}}\Big]_0^8=\frac{96}{5}\pi$

Ex.



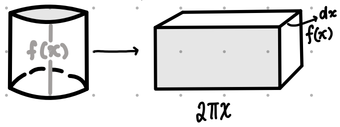
Sol. $A(x)=\pi(2-x^2)^2-\pi(2-x)^2$

$=\pi(4-4x^2+x^4-4+4x-x^2)=\pi(x^4-5x^2+4x)$

$V=\int_0^1\pi(x^4-5x^2+4x)\,dx=\frac{1}{5}x^5-\frac{5}{3}x^3+2x^2\Big]_0^1\pi=\frac{8}{15}\pi$

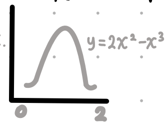
2. Cylindrical Shell

The volume of the solid of rotating about y-axis the region under $y=f(x)$ from a to b is given by $V=\int_a^b 2\pi x f(x)\,dx$.



$dV=2\pi x f(x)\,dx\rightarrow V=\int_a^b 2\pi x f(x)\,dx$

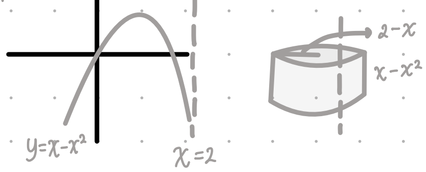
Ex.



Sol. To solve $2x^2-x^3=0$, we have $x=0$ and $x=2$. The volume is

$V=\int_0^2 2\pi x(2x^2-x^3)\,dx=\frac{16}{5}\pi$

Ex.



Sol. The volume is $V=\int_0^1 2\pi(2-x)(x-x^2)\,dx=\frac{\pi}{2}$

Average

Def. The average value of a continuous function f on $[a,b]$ is $f_{avg}=\frac{\int_a^b f(x)dx}{b-a}$.

Thm. MVT for integrals

If f is continuous on $[a,b]$, there is a number $a\leq c\leq b$ such that

$f(c)=f_{avg}=\frac{\int_a^b f(x)dx}{b-a}\rightarrow\int_a^b f(x)\,dx=f(c)(b-a)$

Proof. $F(x)=\int_a^x f(t)\,dt$ is continuous on $[a,b]$ and is differentiable on (a,b)

By FCT

By MVT, $\exists c \in (a,b)$ such that $\frac{\overbrace{F(b)-F(a)}^{\int_a^b f(x)\,dx}}{b-a} = F'(c) = f(c)$

Ex. $\lim_{x\to 0} \frac{\int_x^{\tan x} \sqrt{1+t^3}\,dt}{x^3} =$

Sol. By MVT for integrals, there is a c between x and $\tan x$ such that

$$\underbrace{\int_x^{\tan x} \sqrt{1+t^3}\,dt}_{f(t)} = \underbrace{\sqrt{1+c^3}}_{f(c)}(\tan x - x)$$

As $x \to 0$, we have $\tan x \to 0$. So, $c \to 0$ by Squeeze Theorem,

$$\lim_{x\to 0} \frac{\int_x^{\tan x} \sqrt{1+t^3}\,dt}{x^3} = \lim_{x\to 0} \sqrt{1+c^3} \cdot \frac{\tan x - x}{x^3} \overset{LH}{=} \lim_{x\to 0} \sqrt{1+c^3} \cdot \frac{\overbrace{\sec^2 x - 1}^{\tan^2 x}}{3x^2} = \frac{1}{3}$$