APPLICATION

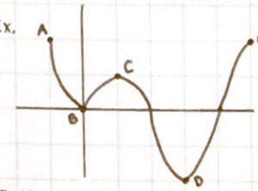
// of derivative

MAX & MIN

Def. Let c be a number in the domain D of a function f. Then f(c) is called absolute maximum value of f on D if $f(c) \geq f(x)$ for all x in D and absolute minimum value if $f(c) \leq f(x)$

Def. The number f(c) is a local maximum value of f if $f(c) \geq f(x)$ when x is near c and local minimum if $f(c) \leq f(x)$

Ex.



$f(x) = 3x^4 - 16x^3 + 18x^2$ [1, 4]

abs. max : f(-1) = 37

abs. min : f(-3) = -27

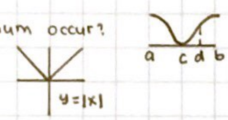
local max : f(1) = 5

local min : f(0) = 0 & f(3) = -27

EXTREME VALUE THEOREM

If f is continuous on a closed interval [a, b], then f attains an absolute maximum value f(c) and an absolute minimum value f(d) for some c, d in [a, b].

Q. Where may the maximum or minimum occur?



FERMET THEOREM

If f has a local maxima or a local minima at c and f'(c) exists, then f'(c) = 0. 0 is a local minima of f, f'(0) doesn't exist.

Def. A critical number of a function f is a number c such that

1. f(c) is defined

2. f'(c) = 0 or f'(c) doesn't exist

Ex. Find the critical number of

(a) $f(x) = x^{\frac{3}{5}}(4-x)$

Sol. $f'(x) = \frac{3}{5}x^{-\frac{2}{5}}(4-x) - x^{\frac{3}{5}} = \frac{4}{5}x^{-\frac{2}{5}}(3-2x)$

It is easy to see that f'(0) doesn't exist and f(0) = 0. Hence, 0 is a critical number of f. To solve f'(x) = 0, we obtain $x = \frac{3}{2}$, $f(\frac{3}{2})$ is defined. $x = \frac{3}{2}$ is a critical number of f.

(b) $g(x) = \frac{3x^{2/3}}{x-1}$

Sol. $g'(x) = \frac{-x-2}{x^{1/3}(x-1)^2}$

So we have g'(0) and g'(1) are not defined. Since g(0) = 0 and g(1) is not defined, x = 0 is a critical number and x = 1 is not a critical number. Next to solve g'(x) = 0, we have x = -2. Since g(-2) is defined, x = -2 is a critical num.

If f has a local maximum or local minimum at c, then c is a critical number.

Proof. If f'(c) does not exist, c is a critical number. If f'(c) exists, by Fermet's Theorem, f'(c) = 0. c is a critical number of f.

Q. If c is a critical number, is f(c) a local maxima, or a local minima?

To find the absolute maximum and absolute minimum, of a continuous function f on [a, b]:

1. Find the critical numbers and evaluate them

2. Evaluate f(a) and f(b)

3. Absolute maximum : largest value found in (1) & (2)
Absolute minimum : smallest value found in (1) & (2)

Ex. Find the absolute maximum & absolute minimum of $f(x) = x^3 - 3x^2 + 1$ $[-\frac{1}{2}, 4]$

Sol. $f'(x) = 3x^2 - 6x$ is defined for $(-\frac{1}{2}, 4)$. To solve f'(x) = 0, we have x = 0 or 2. f(0) = 1, f(2) = -3, $f(-\frac{1}{2}) = \frac{1}{8}$, f(4) = 17. The absolute maximum of f is 17 and absolute minimum of f is -3.

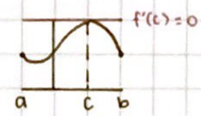
MEAN VALUE

Rolle's Theorem

1. f is continuous on [a, b]

2. f is differentiable on (a, b)

3. f(a) = f(b)



Then, there is a number f(a) = f(b) c such that f'(c) = 0

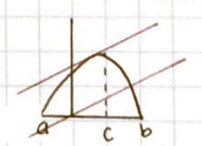
Ex. Prove $x^3 + x - 1 = 0$ has exactly 1 real root

Sol. First, we want to prove there is a root of $x^3 + x - 1 = 0$. Set $f(x) = x^3 + x - 1$, $f(0) \overset{<0}{=} -1$, $f(1) \overset{>1}{=} 1$. By intermediate value theorem, there is a number 0 < c < 1 such that f(c) = 0. Assume there are 2 roots $\alpha < \beta$ of f(x) = 0. By Rolle's Theorem, there is a $\alpha < \gamma < \beta$ such that $f'(\gamma) = 0$. But $f'(x) = 3x^2 + 1 \geq 1$ we get a contradiction. Therefore, $x^3 + x - 1 = 0$ has exactly one root.

Mean Value Theorem

1. f is a continuous on [a, b]

2. f is differentiable (a, b)



a < b < c

Then, there is a number $c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

$f(b) - f(a) = f'(c)(b-a)$

Generalized Mean Value Theorem

If f and g are both continuous on [a, b] and differentiable on (a, b), then there is a $c \in (a, b)$ such that $(f(b)-f(a))g'(c) = (g(b)-g(a))f'(c)$

Proof. Define $g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$. Since f is continuous on [a, b] and differentiable on (a, b). g is continuous on [a, b] and differentiable on (a, b). g(a) = 0 and g(b) = 0. By Rolle's Theorem, there is a number $c \in (a, b)$ such that $0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$. Therefore $f'(c) = \frac{f(b)-f(a)}{b-a}$

Ex. f(0) = -3, $f'(x) \leq 5$ for all x. How large can f(2) possible be?

Sol. By $f'(x) \leq 5$ for all x, we know f is continuous on [0, 2] and is differentiable on (0, 2). By MVT, there is a number $c \in (0, 2)$ such that $f(2) - f(0) = f'(c)(2-0) \leq 5(2-0) \Rightarrow f(2) \leq f(0) + 10 = 7$

Thm. If $f'(x) \equiv 0$ on (a, b), then $f(x) \equiv C$ on (a, b) where C is a const.

Proof. Choose any $a<x_1<x_2<b$. By MVT, there is a number $x_0 \in (x_1, x_2)$ such that $f(x_2)-f(x_1) = f'(x_0)(x_2-x_1)$ since $f'(x) \equiv 0$ on (a,b), we have $f(x_1)=f(x_2)$. Since x_1 and x_2 are arbitrary, $f(x) \equiv C$ on (a,b) where c is a constant.

Corollary. If $f'(x) = g'(x)$, then $f(x) = g(x)+C$, c is a constant

Proof. Set $h(x) = f(x) - g(x)$, $h'(x) = f'(x) - g'(x) = 0$. Therefore, $f(x) - g(x) = h(x) = C$ where c is a constant.

Known. $f' = g'$, $g \Rightarrow f = g + C$

G R A P H A F F E C T

1. 1^{st} derivative: increasing/decreasing

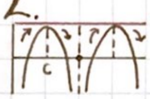
2. 2^{nd} derivative: concavity/inflection point

3. Classify the critical number
   - The 1^{st} derivative test
   - The 2^{nd} derivative test

Increasing / decreasing

1. If $f'(x)>0$ on the interval I, then f is strictly increasing on I.

2. If $f'(x)<0$ on the interval I, then f is strictly decreasing on I.



$f'(1)$ does not exist

How to find where f is increasing & where f is decreasing:

1. Compute $f'(x)$

2. Find the point such that $f'(x)=0$ or $f'(x)$ does not exist

3. Assume $-\infty < a < b \leq -\infty$ f and in (2) choose $d \in (a,b)$

   i. If $f(d)>0$, f is increasing on (a,b)

   ii. If $f(d)<0$, f is decreasing on (a,b)

Thm. Suppose f is continuous on $[a,b]$, $f(a)=f(b)=0$ and $f(x) \neq 0$ on (a,b). If $f(d)<0$ / $f(d)>0$ on (a,b) then $f(x)<0$ / $f(x)>0$ for all $x \in (a,b)$.

Ex. $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ where f is increasing? where f is decreasing?

Sol. $f'(x) = 12x^3 - 12x^2 - 24x = 12x(x-2)(x+1)$ is defined for all $x \in \mathbb{R}$. To solve $f'(x)=0$, we have $x = 0, 2, -1$. For $x>2$, $f'(x) = 12\overset{+}{x}(\overset{+}{x-2})(\overset{+}{x+1})>0$. For $0<x<2$, $f'(x) = 12\overset{+}{x}(\overset{-}{x-2})(\overset{+}{x+1})<0$. For $-1<x<0$, $f'(x) = 12\overset{-}{x}(\overset{-}{x-2})(\overset{+}{x+1})>0$. For $x<-1$, $f'(x) = 12\overset{-}{x}(\overset{-}{x-2})(\overset{-}{x+1})<0$. Therefore, f is increasing on $(2,\infty)$ and $(-1,0)$ and is decreasing on $(0,2)$ and $(-\infty,-1)$.

Ex. $f(x) = \frac{\ln|x|}{x}$, where is f increasing? where is f decreasing?

Sol. For $x>0$, $f(x) = \frac{\ln x}{x}$; $f'(x) = \frac{1-\ln x = 0}{x^2}$

To solve $f'(x)=0$, we have $x=e$.

For $x<0$, $f(x) = \frac{\ln(-x)}{x}$; $f'(x) = \frac{1-\ln(-x)}{x^2}$

To solve $f'(x)=0$, we have $x=-e$

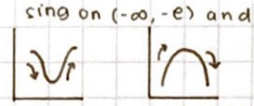
$f'(e^2) = \frac{1-\ln e^2}{e^4} = -\frac{1}{e^4} < 0$

$f'(\sqrt{e}) = \frac{1-\ln\sqrt{e}}{e} = \frac{1}{2e} > 0$

$f'(-\sqrt{e}) = \frac{1-\ln\sqrt{e}}{e} = \frac{1}{2e} > 0$

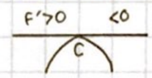
$f'(-e^2) = \frac{1-\ln e^2}{e^4} = \frac{-1}{e^4} < 0$

Therefore, f is increasing on $(-e,0)$ and $(0,e)$ and is decreasing on $(-\infty,-e)$ and (e,∞).

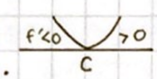


The first derivative test

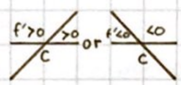
// c is a critical number

1. 

f has local maxima at c

2. 

f has local minima at c

3. 

f has no local extremum at c

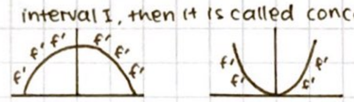
Ex. Find the local max & min of $f(x) = \frac{x^2}{(x+1)^2}$

Sol. $f'(x) = \frac{x^2(x+3)}{(x+1)^3}$ since $f(-1)$ is not defined, $x=-1$ is not a critical number. To solve $f'(x)=0$, we have $x=0$ and -3.

$f'(1) = \frac{1}{2}$; $f'(-\frac{1}{2}) = \cdots > 0$; $f(-2) = -4$; $f(-4) = \frac{16}{27}$

Therefore, f has a local max at $x=-3$ and there is no local min.

Def. If the graph of f lies above / below all of its tangents on an interval I, then it is called concave upward / downward on I.

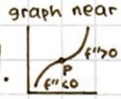


Concavity Test

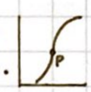
If $f''(x)>0$ / $f''(x)<0$ for all x on an interval I, then the graph of f is concave upward / downward on I.

Def. A point $P(c, f(c))$ on a curve $y=f(x)$ is called an inflection point if $\frac{f''<0 \quad >0}{c}$ or $\frac{f''>0 \quad <0}{c}$

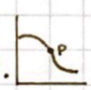
The graph near $P(c, f(c))$

1. 

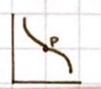
f'': $- \to +$
$f'(c)>0$

2. 

f'': $+ \to -$
$f'(c)>0$

3. 

f'': $- \to +$
$f'(c)<0$

4. 

f'': $+ \to -$
$f'(c)<0$

The Second Derivative Test

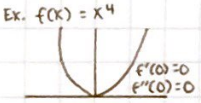
Suppose f'' is continuous at c

1. $f'(c)=0$; $f''(c)>0$
   f has a local minimum at c

2. $f'(c)=0$; $f''(c)<0$
   f has a local maximum at c

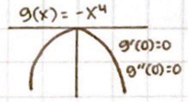
3. $f'(c)$ and $f''(c)$
use 1^{st} derivative test

Ex. $f(x) = x^4$



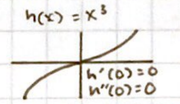
f has a local min at 0

$g(x) = -x^4$



local max

$h(x) = x^3$



has no local extremum

Ex. Discuss the curve $y = x^4 - 4x^3$ with respect to concavity, point of inflection, local minimum or local maximum

Sol. $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$; $f''(x) = 12x^2 - 24x = 12x(x-2)$

Set $f(x) = x^4 - 4x^3$ $\quad$ $+ \underset{0}{\sim} - \underset{2}{\sim} +$

For $x>2$ and $x<0$, $f''(x) > 0$. For $0<x<2$, $f''(x)<0$. So, the graph of $y = x^4 - 4x^3$ is concave upward for $x>0$ and $x<0$ and is concave downward for $0<x<2$. Also $(0, f(0)) = (0,0)$ and $(2, f(2)) = (2,-16)$ are the points of inflection. Since $f'(x)$ is defined for all x, the critical number f is $x=0$ and $x=1$ by solving $f'(x)=0$. Since $f'(1)=0$ and $f''(1)<0$, f has a local minimum at $x=3$. Since $f''(0)=0$, we cannot use the second derivative test. Since $f'(x)<0$ for $x<0$ and $0<x<3$, f has no local extremum at $x=0$.

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Rule

1. f, g : differentiable function with $g'(x) \neq 0$ on an open interval I that contains a

2. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \pm\infty$ or $\lim_{x\to a} g(x) = \pm\infty$.
Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ // If limit exists (including $\pm\infty$)

case

1. Indeterminate form // $\frac{0}{0}$

Ex. find $\lim_{x\to 1} \frac{\ln x}{x-1}$

Sol. $\lim_{x\to 1} \ln x = 0$ and $\lim_{x\to 1} x-1 = 0$. Use the l'Hospital's rule,
$\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1$

2. Indeterminate form // $\frac{\infty}{\infty}$

Ex. Find $\lim_{x\to\infty} \frac{e^x}{x^2}$

Sol. $\lim_{x\to\infty} \frac{e^x}{x^2} \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{LH}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty$

3. $\lim_{x\to a} f(x)g(x)$: Indeterminate
$\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = \pm\infty$
$\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = 0$
If $\lim_{x\to a} f(x) = 0$ for m $\lim_{x\to a} f(x) \cdot g(x) = \lim_{x\to a} \frac{f(x) \to 0}{\frac{1}{g(x)} \to 0}$

Ex. $\lim_{x\to 0^+} x\ln x$

Sol. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x \to -\infty}{\frac{1}{x} \to +\infty} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} -x = 0$

4. $\lim_{x\to a} f(x) - g(x)$ with $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = +\infty$
Indeterminate form // $\infty - \infty$

Ex. Find $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

Sol. As $x\to 1^+$, $\ln x \to 0^+$, and $x-1 \to 0^+$. So, $\frac{1}{\ln x} \to +\infty$ and $\frac{1}{x-1} \to +\infty$ as $x \to 1^+$. $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+}\left(\frac{x-1-\ln x \to 0}{\ln x(x-1) \to 0}\right) \overset{LH}{=} \lim_{x\to 1^+} \frac{1-\frac{1}{x} \to 0}{\frac{1}{x}(x-1)+\ln x \to 0} = \lim_{x\to 1^+} \frac{x-1}{x-1+x\ln x}$
$\overset{LH}{=} \lim_{x\to 1^+} \frac{1}{1+\ln x+1} = \frac{1}{2}$

5. $\lim_{x\to\infty} [f(x)]^{g(x)}$
Indeterminate form // $1^\infty, \infty^0, 0^0$

Ex. $\lim_{x\to 0^+} (1+\sin 4x)^{\cot x}$

Sol. $(1+\sin 4x)^{\cot x} = e^{\cot x \cdot \ln(1+\sin 4x)}$

$\lim_{x\to 0^+} \cot x \cdot \ln(1+\sin 4x) = \lim_{x\to 0^+} \frac{\ln(1+\sin 4x)}{\tan x} \overset{LH}{=} \lim_{x\to 0^+} \frac{4\cos 4x}{1+\sin 4x} \cdot \frac{1}{\sec^2 x} = 4$
$\lim_{x\to 0^+} (1+\sin 4x)^{\cot x} = \lim_{x\to 0^+} e^{\cot x \ln(1+\sin 4x)} = e^{\lim_{x\to 0^+} \cot x \ln(1+\sin 4x)} = e^4$

Sol. Set $y = (1+\sin 4x)^{\cot x}$
$\ln y = \cot x \cdot \ln(1+\sin 4x)$
$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \cot x \cdot \ln(1+\sin 4x) = 4$
$\ln(\lim_{x\to 0^+} y) = 4 \Rightarrow \lim_{x\to 0^+} y = e^4$

Ex. $\lim_{x\to 0^+} x^x$

Sol. Set $y = x^x$; $\ln y = x\ln x$
$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x \to -\infty}{\frac{1}{x} \to +\infty} \overset{LH}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$
So, $\lim_{x\to 0^+} \ln y = 0 \Rightarrow \lim_{x\to 0^+} y = 1$

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Slant Asymptote // 歪斜漸近線

Def. The line $y = mx+b$, $m \neq 0$ is called a slant asymptote if
$\lim_{x\to\infty} [f(x) - (mx+b)] = 0$

In fact, $m = \lim_{x\to\infty} \frac{f(x)}{x}$ and $b = \lim_{x\to\infty} [f(x) - mx]$. If $\lim_{x\to\infty} \frac{f(x)}{x}$ does not exist or 0, or $= \pm\infty$, there is no slant asymptote.

$y = f(x)$

1. Find the domain of f

2. Intercept: y-intercept : $(0, f(0))$
x-intercept : $(x_0, 0)$; where $f(x_0) = 0$

3. Symmetry
Check: (i) $f(x) = f(-x)$
(ii) $f(x) = -f(-x)$
(iii) $f(x+p) = f(x)$; $p \neq 0$ ⇒ period of f

4. Asymptotes
(i) vertical asymptotes : $f(c)$ is not defined.
Check $\lim_{x\to c^+} f(x) \overset{?}{=} \pm\infty$ or $\lim_{x\to c^-} f(x) \overset{?}{=} \pm\infty$
(ii) horizontal asymptotes
check $\lim_{x\to\infty} f(x)$ & $\lim_{x\to -\infty} f(x)$
(iii) Slant asymptotes
check $\lim_{\substack{x\to\infty \\ x\to -\infty}} \frac{f(x)}{x}$ exists and $\neq 0$. Set $m = \lim_{\substack{x\to\infty \\ x\to -\infty}} \frac{f(x)}{x}$, $b = \lim_{x\to\infty} (f(x) - mx)$
⇒ The line $y = mx+b$ is a slant asymptote.

Next, we compute f'

5. Increase / decrease

6. Local max & local min (1^{st} derivative test)

Next, we compute f''

7. Concavity & point of inflection

8. Sketch the graph

Ex. $f(x) = \ln x - \ln(\ln x)$

Sol. 1. Find the domain of f
To define f, we require $x>0$ and $\ln x > 0$, $x>1$. The domain of f is $x>1$. Since $x>1$, there is no y-intercept. To obtain x-intercept, we need to $\ln x - \ln(\ln x) = 0$. But it is difficult.

2. Asymptote

Since $\lim_{x\to 1^+} \ln x - \ln(\ln x) = +\infty$, we have $x=1$ is a vertical asymptote. $\lim_{x\to\infty} \ln x - \ln(\ln x) = \lim_{x\to\infty} \ln\left(\frac{x}{\ln x}\right) = \ln\left(\lim_{x\to\infty}\frac{x}{\ln x}\right) \overset{LH}{=} \ln\left(\lim_{x\to\infty}\frac{1}{\frac{1}{x}}\right) = \infty$. There is no horizontal asymptote. $\lim_{x\to\infty}\frac{\ln x - \ln(\ln x)}{x} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{x}-\frac{1}{x\ln x}}{1} = \lim_{x\to\infty}\frac{\ln x - 1}{x\ln x} \overset{LH}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{\ln x + 1} = \lim_{x\to\infty}\frac{1}{x\ln x + x} = 0$. There is no slant asymptote.

3. Increase / decrease ; local max & local min

$f'(x) = \frac{1}{x} - \frac{1}{x\ln x} = \frac{\ln x - 1}{x\ln x}$

For $x>1$, $f'(x)$ is defined. To solve $\ln x - 1 = 0$ for $x>1$, we have $x=e$. For $x>e$, $\ln x > 1$, and $f'(x)>0$. For $1<x<e$, $\ln x<1$, and $f'(x)<0$. Then we have $f(x)$ is increasing for $x>e$ and is decreasing for $1<x<e$. f has a local / absolute minimum $f(e)=1$ at $x=e$.

4. Concavity & inflection

$f''(x) = \frac{-[(\ln x)^2 - \ln x - 1]}{(x\ln x)^2}$

Set $t=\ln x$. To solve t^2-t-1 for $t>0$, we have $t=\frac{1+\sqrt{5}}{2}$ and $x=e^{\frac{1+\sqrt{5}}{2}}$. For $x>e^{\frac{1+\sqrt{5}}{2}}$, $(\ln x)^2-\ln x-1<0$

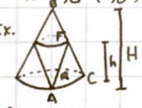
The graph of f is concave upward for $1<x<e^{\frac{1+\sqrt{5}}{2}}$ and is concave downward for $x>e^{\frac{1+\sqrt{5}}{2}}$. Since f'' change sign at $x=e^{\frac{1+\sqrt{5}}{2}}$, $\left(e^{\frac{1+\sqrt{5}}{2}}, f\left(e^{\frac{1+\sqrt{5}}{2}}\right)\right) = \left(e^{\frac{1+\sqrt{5}}{2}}, \frac{1+\sqrt{5}}{2} - \ln\left(\frac{1+\sqrt{5}}{2}\right)\right)$ is a point of inflection.

O · P · T · I · M · A · L ·

Ex.  $V = 1L = 1000\ cm^3$ Minimize the cost of the metal to manufacture the can

Sol. Set h cm is the height of the can and r cm is the radius of the bottom of the can. To minimize the cost, we need to minimize the surface of the can. The area of the surface is $2\pi r^2 + 2\pi r h$. Also the volume $\pi r^2 h = 1000 \to h = \frac{1000}{\pi r^2}$. So, the area of the surface is $A(r) = 2\pi r^2 + 2\pi r\cdot\frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$. $A'(r) = 2\left[2\pi r - \frac{1000}{r^2}\right] = \frac{4}{r^2}[\pi r^3 - 500]$.

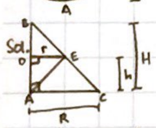
To solve $A'(r)=0$, we have $r=\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$. For $r>\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$, $A'(r)>0$. For $0<r<\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$, $A'(r)<0$. Therefore, when $r=\left(\frac{500}{\pi}\right)^{\frac{1}{3}}$, $h \overset{=2r}{=} \frac{1000}{\pi}\left(\frac{500}{\pi}\right)^{-\frac{2}{3}} = 2\left(\frac{500}{\pi}\right)^{\frac{1}{3}} = 2\cdot\frac{(500/\pi)\left(\frac{500}{\pi}\right)^{-\frac{2}{3}}}{(500/\pi)^{1/3}}$ the cost is minimized.

Ex. 

H, R : height and radius of large cone

h, r : height and radius of small cone

Find h that maximize the volume of the small cone

Sol. 

$\frac{r}{R} = \frac{H-h}{H} \to r = \frac{H-h}{H}R$

The volume of the small cone is $V(h) = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{H-h}{H}\right)^2 R^2 h = \frac{\pi R^2}{3H^2}(H-h)^2 h$, $0<h<H$.

$V'(h) = \frac{\pi R^2}{3H^2}\left[-2(H-h)h + (H-h)^2\right] = \frac{\pi R^2}{3H^2}(H-h)(H-3h)$

For $0<h<H$, to solve $V'(h)=0$, we have $h=\frac{H}{3}$. For $0<h<\frac{H}{3}$, $V'(h)>0$. For $\frac{H}{3}<h<H$, $V'(h)<0$. When $h=\frac{H}{3}$, $r=\frac{2}{3}R$, the max value.

A · N · T · I · D · E · R · I · V · A · T · I · V · E ·

Known. $F'=f$. What is the function F?

Ex. $v(t)$: velocity function $\quad x'(t) = v(t)$; $x(t)$?

$x(t)$: position function

Def. A function F is called an antiderivative of f on an open interval I if $F'(x)=f(x)$ for all x in I.

Thm. If $F(x)$ is an antiderivative of f, then the most general antiderivative of f on I is $F(x)+C$ where C is a constant.

Proof. $(F(x)+C)' = F'(x) + (C)' = f(x) + 0 = f(x)$ by the fact [$F(x)$ is an antiderivative of f]. So $F(x)+C$ is antiderivative of f.

Assume $G(x)$ is antiderivative of f. Then $(G(x)-F(x))' = G'(x) - F'(x) = f - f = 0$, for all x in I. Therefore is a constant C such that $G(x) - F(x) = C$.

C · O · N · C · L · U · S · I · O · N ·

If we can find a function $F(x)$ is an antiderivative of f, then any antiderivative of f is the form $F(x)+C$

Functions	Antiderivative
x^n $(n \neq -1)$	$\frac{1}{n+1}x^{n+1}$
x^{-1}	$\ln\lvert x\rvert$
e^x	e^x
b^x	$\frac{1}{\ln b}\cdot b^x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x\tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x$
$\frac{1}{1+x^2}$	$\tan^{-1}x$

Ex. Find all g such that $g'(x) = 4\sin x + \frac{2x^5-\sqrt{x}}{x}$

Sol. $g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$. So, $g(x) = 4(-\cos x) + \frac{2}{4+1}x^{4+1} - \frac{1}{-\frac{1}{2}+1}x^{-\frac{1}{2}+1} + C$
$= -4\cos x + \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + C$ where C is a constant.

Differential equation : An equation that involves the derivative of a function.

Ex. Find f if [$f'(x) = e^x + 20(1+x^2)^{-1}$; $f(0)=-2$] : Initial value function initial condition

Sol. The antiderivative of f' is $e^x + 20\tan^{-1}x + C$. So the form of $f(x)$ is $e^x + 20\tan^{-1}x + C$. Since $f(0)=-2$, we have
$-2 = f(0) = e^0 + 20\tan^{-1}0 + C \Rightarrow C = -3$

Therefore, the particular solution is $f(x) = e^x + 20\tan^{-1}x - 3$.

Ex. u, v : population density of 2 species

$\frac{du}{dt} = u(1-u-kv)$; $\frac{dv}{dt} = v(1-v-hu)$; Two species competition system.