Calculus Quiz 2 *2020/10/28*

Department: ____________ Student ID: ____________ Name: ____________

1. Find the derivative of the following functions.(You need not to simplify your answer.)

   (a) $\dfrac{\mathrm{d}^7}{\mathrm{d}x^7}(\dfrac{\sin^2 x}{1+\cot x}+\dfrac{\cos^2 x}{1+\tan x})$ (10 points)

   (b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\csc^{-1}(\dfrac{\sqrt{1+x^2}}{x}))$ (10 points)

   (c) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\log_{2^x}(\log_{x^2}\sec x))$ (15 points)

   Solution:

   (a) Primarily, we can simplify the formula because direct differentiation is difficult.

$$\begin{aligned}\frac{\mathrm{d}^7}{\mathrm{d}x^7}(\frac{\sin^2 x}{1+\cot x}+\frac{\cos^2 x}{1+\tan x}) &= \frac{\mathrm{d}^7}{\mathrm{d}x^7}(\frac{\sin^3 x}{\sin x+\cos x}+\frac{\cos^3 x}{\sin x+\cos x})\\ &= \frac{\mathrm{d}^7}{\mathrm{d}x^7}(\frac{\sin^3 x+\cos^3 x}{\sin x+\cos x})\\ &= \frac{\mathrm{d}^7}{\mathrm{d}x^7}(\sin^2 x-\sin x\cos x+\cos^2 x)\\ &= \frac{\mathrm{d}^7}{\mathrm{d}x^7}(1-\frac{1}{2}\sin 2x)\\ &= 64\cos 2x\end{aligned}$$

   (b) Using the chain rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}(\csc^{-1}(\frac{\sqrt{1+x^2}}{x})) = \frac{-1}{\frac{\sqrt{1+x^2}}{x}\cdot\sqrt{(\frac{\sqrt{1+x^2}}{x})^2-1}}\cdot\frac{\frac{2x}{2\sqrt{1+x^2}}\cdot x-\sqrt{1+x^2}\cdot 1}{x^2} = \frac{1}{1+x^2}$$

   (c) Let $y=\log_{2^x}(\log_{x^2}\sec x)$, we want to find y'.

$$\begin{aligned}(2^x)^y &= \log_{x^2}\sec x\\ \frac{\mathrm{d}}{\mathrm{d}x}(2^{xy}) &= \frac{\mathrm{d}}{\mathrm{d}x}(\log_{x^2}\sec x)\\ 2^{xy}\cdot\ln 2\cdot(y+xy') &= \frac{\mathrm{d}}{\mathrm{d}x}(\log_{x^2}\sec x)\\ y' &= \frac{1}{x}(\frac{1}{\ln 2\cdot\log_{x^2}\sec x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\log_{x^2}\sec x)-\log_{2^x}(\log_{x^2}\sec x))\end{aligned}$$

   Let $u=\log_{x^2}\sec x$, we want to find u'.

$$\begin{aligned}(x^2)^u &= \sec x\\ \frac{\mathrm{d}}{\mathrm{d}x}(x^{2u}) &= \frac{\mathrm{d}}{\mathrm{d}x}(\sec x)\\ \frac{\mathrm{d}}{\mathrm{d}x}(e^{2u\ln x}) &= \sec x\tan x\\ e^{2u\ln x}\cdot(2u'\ln x+\frac{2u}{x}) &= \sec x\tan x\\ u' &= \frac{1}{\ln x}(\frac{1}{2}\tan x-\frac{\log_{x^2}\sec x}{x})\end{aligned}$$

   So, we can conclude

$$y'=\frac{1}{x}(\frac{1}{\ln 2\cdot\log_{x^2}\sec x}\cdot\frac{1}{\ln x}(\frac{1}{2}\tan x-\frac{\log_{x^2}\sec x}{x})-\log_{2^x}(\log_{x^2}\sec x))$$

2. Show that

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx+n\theta)$$

where a and b are positive numbers, $r^2 = a^2 + b^2$, and $\theta = \tan^{-1}(\frac{b}{a})$.
(Hint: You can use the principle of mathematical induction to prove it.) (20 points)

Solution:

First, show that $n = 1$ holds,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(e^{ax}\sin bx) &= ae^{ax}\sin bx + be^{ax}\cos bx \\ &= \sqrt{a^2+b^2}e^{ax}(\sin bx\frac{a}{\sqrt{a^2+b^2}} + \cos bx\frac{b}{\sqrt{a^2+b^2}})\end{aligned}$$

Let $r = \sqrt{a^2+b^2}$, $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\theta = \frac{b}{\sqrt{a^2+b^2}}$, and $\theta = \tan^{-1}(\frac{b}{a})$.

So,

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^{ax}\sin bx) = re^{ax}(\sin bx\cos\theta + \cos bx\sin\theta) = re^{ax}\sin(bx+\theta).$$

Second, suppose that $n = k$ holds, that is,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx+n\theta)$$

We want to show that $n = k+1$ holds,

$$\begin{aligned}\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(e^{ax}\sin bx) &= \frac{\mathrm{d}}{\mathrm{d}x}(r^n e^{ax}\sin(bx+n\theta)) \\ &= ar^n e^{ax}\sin(bx+n\theta) + br^n e^{ax}\cos(bx+n\theta) \\ &= \sqrt{a^2+b^2}r^n e^{ax}(\sin(bx+n\theta)\frac{a}{\sqrt{a^2+b^2}} + \cos(bx+n\theta)\frac{b}{\sqrt{a^2+b^2}}) \\ &= r^{n+1}e^{ax}\sin(bx+(n+1)\theta)\end{aligned}$$

By the principle of mathematical induction, the formula holds for all $n \in \mathbb{N}$.

3. The curve $(x^2+y^2)^2 = x^2 - y^2$ is called a *lemniscate*. Find all points of the curve at which the tangent line is horizontal. (15 points)

Solution:

By implicit differentiation, we have

$$2(x^2+y^2)(2x+2yy') = 2x - 2yy'$$

In order to find horizontal tangent line, let $y' = 0$, and we have

$$4x(x^2+y^2) = 2x$$

That is, $x = 0$ or $x^2+y^2 = \frac{1}{2}$. Slove x, y subject to $(x^2+y^2)^2 = x^2-y^2$, we have

$$(x,y) = (0,0), (\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}), (\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}), (-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}), (-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4})$$

Ruling out the first improper case, we have

$$(x,y) = (\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}), (\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4}), (-\frac{\sqrt{6}}{4}, \frac{\sqrt{2}}{4}), (-\frac{\sqrt{6}}{4}, -\frac{\sqrt{2}}{4})$$

4. Let $f'(x) = \dfrac{1}{\sqrt[3]{1-f^3(x)}}$ and $f''(x) = \dfrac{1}{f(x)-1}$, Find $g''(x)$, where $g(x)$ is the inverse function of $f(x)$. (15 points)

Solution:

By the definition of inverse function $g(f(x)) = x$, we have

$$\begin{aligned} g'(f(x))f'(x) &= 1 \\ g'(f(x)) &= \frac{1}{f'(x)} \\ g'(x) &= \frac{1}{f'(g(x))} \end{aligned}$$

Similarly, the second derivative is

$$\begin{aligned} g''(f(x))(f'(x))^2 + g'(f(x))f''(x) = 0& \\ g''(f(x)) &= -\frac{g'(f(x))f''(x)}{(f'(x))^2} \\ g''(f(x)) &= -\frac{f''(x)}{(f'(x))^3} \\ g''(x) &= -\frac{f''(g(x))}{[f'(g(x))]^3} \end{aligned}$$

So,

$$g''(x) = -\frac{\dfrac{1}{f(g(x))-1}}{\left(\dfrac{1}{\sqrt[3]{1-f^3(g(x))}}\right)^3} = -\frac{1-f^3(g(x))}{1-f(g(x))}$$

Since $f(g(x)) = x$, we have

$$g''(x) = \frac{1-x^3}{1-x} = 1 + x + x^2$$

5. If $P(a, a^2)$ is any first-quadrant point on the parabola $y = x^2$, let Q be the point where the normal line at P intersects the parabola again. For what values of a does the segment PQ have the shortest length? (15 points)

Solution:

The slope of the tangent line is the derivative of the parabola at $x = a$ which is $2a$. Thus, the slope of the normal line is the opposite of reciprocal of $2a$, which is $-\dfrac{1}{2a}$. Now using the point-slope formula, we find the equation of the normal line

$$y = -\frac{1}{2a}(x-a) + a^2$$

To find the coordinates of Q, we solve the system of the parabola and the normal line, and use substitution method,

$$
\begin{aligned}
x^2 &= -\frac{1}{2a}(x-a)+a^2 \\
x^2-a^2 &= -\frac{1}{2a}(x-a) \\
x+a &= -\frac{1}{2a} \\
x &= -\frac{1}{2a}-a \\
y &= -\frac{1}{2a}(-\frac{1}{2a}-2a-a)+a^2 \\
y &= a^2+1+\frac{1}{4a^2}
\end{aligned}
$$

Hence the coordinates of $Q(-a-\frac{1}{2a}, a^2+1+\frac{1}{4a^2})$.
Now we find the segment PQ, and let the segment be $D(a)$.
Then, using the distance formula, we have

$$
\begin{aligned}
D^2(a) &= (-a-\frac{1}{2a}-a)^2+(a^2+1+\frac{1}{4a^2}-a^2)^2 \\
&= (-2a-\frac{1}{2a})^2+(1+\frac{1}{4a^2})^2 \\
&= 4a^2+3+\frac{3}{4a^2}+\frac{1}{16a^4}
\end{aligned}
$$

By Fermat's Theorem, if $D^2(a)$ has a local maximum or minimum at c, which is critical number, and if $\frac{\mathrm{d}}{\mathrm{d}a}D^2(c)$ exists,then $\frac{\mathrm{d}}{\mathrm{d}a}D^2(c)=0$, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}a}D^2(a) &= 0 \\
8a-\frac{3}{2a^3}-\frac{1}{4a^5} &= 0 \\
32a^6-6a^2-1 &= 0 \\
(2a^2-1)(4a^2+1)^2 &= 0 \\
a &= \pm\frac{1}{\sqrt{2}}
\end{aligned}
$$

Since we know a is a positive real number, we get $a=\frac{1}{\sqrt{2}}$.