Caculus Quiz 1 *2020/10/7*

Department: ________________ ID Number: ________________ Name: ________________

1. (? points) Compute each of the following limits if it exists or explain why it doesn't exist.
   (a) $\lim_{x\to+\infty}\frac{\sqrt{4x^4+1}}{e^{x^2}}$ (b) $\lim_{x\to 0}(\sin^2 x)2^{\cos(\frac{1}{x})}$ (c) $\lim_{x\to\infty}\frac{(\sqrt{x}+x)^2}{1+x\sqrt{x}}$ (d) $\lim_{x\to\infty}\sin(\ln(\frac{1}{x}))$

Solution:

(a) (Method 1)
Exponential functions grow asymptotically faster than algebraic functions.
Since $\sqrt{4x^4+1}$ grows asymptotically slower than e^{x^2} as x approaches ∞, $\lim_{x\to\infty}e^{-x^2}\sqrt{4x^4+1}=0$.
(Method 2)
For every $x>1$,

$$0\le\frac{\sqrt{4x^4+1}}{e^{x^2}}\le\frac{4x^2}{e^{x^2}}$$

and since $\lim_{x\to+\infty}\frac{4x^2}{e^{x^2}}\overset{(\frac{\infty}{\infty})}{=}\lim_{x\to+\infty}\frac{4}{e^{x^2}}=0$, we see that by Squeeze Theorem, $\lim_{x\to+\infty}\frac{\sqrt{4x^4+1}}{e^{x^2}}=0$.

(b) We have the inequalities,

$$-1\le\cos\left(\frac{1}{x}\right)\le 1\Longrightarrow\frac{1}{2}=2^{-1}\le\cos\left(\frac{1}{x}\right)\le 2^1=2,\ x\ne 0$$

where we have used that for $a\ge b$, $2^a\ge 2^b$ (this follows from the fact that the derivative of 2^x is always positive, so the function is increasing). Since $\sin^2(x)\ge 0$, we get the inequalities

$$\frac{1}{2}\sin^2(x)\le\sin^2(x)\cos\left(\frac{1}{x}\right)\le 2\sin^2(x),\ x\ne 0$$

since $\lim_{x\to 0}\frac{1}{2}\sin^2(x)=0$ and $2\frac{1}{2}\sin^2(x)=0$, by Squeeze Theorem, we find $\lim_{x\to 0}(\sin^2 x)2^{\cos(\frac{1}{x})}=0$.

(c) (Method 1)

$$\begin{aligned}\lim_{x\to\infty}\frac{(\sqrt{x}+x)^2}{1+x\sqrt{x}}&=\lim_{x\to\infty}\frac{x+2x\sqrt{x}+x^2}{1+x\sqrt{x}}\\&=\lim_{x\to\infty}\left(\frac{x+2x\sqrt{x}+x^2}{1+x\sqrt{x}}\cdot\frac{x^{-2}}{x^{-2}}\right)\\&=\lim_{x\to\infty}\frac{\frac{1}{x}+\frac{2}{x^{\frac{1}{2}}}+1}{\frac{1}{x^2}+\frac{1}{x^{\frac{1}{2}}}}\end{aligned}$$

Note that $\lim_{x\to\infty}\left(\frac{1}{x}+\frac{2}{x^{\frac{1}{2}}}+1=1\right)$, while $\frac{1}{x^2}+\frac{1}{x^{\frac{1}{2}}}=0$.
Hence, the limit is of the form $\frac{1}{0}$, which implies that the limit is either ∞ or $-\infty$. Since both the numerator and denominator a positive, then

$$\lim_{x\to\infty}\frac{(\sqrt{x}+x)^2}{1+x\sqrt{x}}=+\infty$$

(Method 2)

$$\begin{aligned}
\lim_{x\to\infty} \frac{(\sqrt{x}+x)^2}{1+x\sqrt{x}} &= \lim_{x\to\infty} \frac{x+2x\sqrt{x}+x^2}{1+x\sqrt{x}} \\
&= \lim_{x\to\infty} \left(\frac{x+2x\sqrt{x}+x^2}{1+x\sqrt{x}} \cdot \frac{x^{\frac{-3}{2}}}{x^{\frac{-3}{2}}} \right) \\
&= \lim_{x\to\infty} \frac{\frac{1}{\sqrt{x}}+2+\sqrt{x}}{\frac{1}{x\sqrt{x}}+1} = \infty
\end{aligned}$$

because the numerator goes to ∞ and the denominator goes to 1.

(d) Since $\lim_{x\to\infty} \frac{1}{x} = 0$, the above is equivalent to $\lim_{y\to 0^+} \sin(\ln(y))$. But $\lim_{y\to 0^+} \ln(y) = -\infty$, so the limit we're considering is equivalent to $\lim_{t\to -\infty} \sin t$. This limit **does NOT exist**, since the value of $\sin t$ oscillates infinitely often between -1 and 1 as t increases without bound, and does not approach a value.

2. (? points) Consider the function $h(x) = e^{(-e^x)} - 2$.

(a) Find the domain and range of h.

(b) Find the equations of all vertical asymptotes of h, or explain completely why none exist. (As justification for each asymptote $x = a$, calculate both the one-sided limits $\lim_{x\to a^-} h(x)$ and $\lim_{x\to a^+} h(x)$ with reasoning.)

(c) Find the equations of all horizontal asymptotes of $h(x)$, or explain why none exist. Justify using limit computations.

(d) It is a fact that $h(x)$ is one-to-one (which you do not have to prove). Find an expression for $h^{-1}(x)$, the inverse of $h(x)$.

Solution:

(a) The domain of h is all of $\mathbb{R} = (-\infty, \infty)$, because h is a composition of exponential and linear functions, all of whose domains are $\mathbb{R}$.
To find the range of h:
(Method 1)
Range of e^x is $(0, \infty)$, range of $-e^x$ is $(-\infty, 0)$, range of e^{-e^x} is $(0, 1)$, range of $e^{-e^x} - 2$ is $\boxed{(-2, -1)}$.
(Method 2)
From part d, $h^{-1}(x) = \ln(-\ln(x + 2))$, and the range of h is the domain of h^{-1}. For $h^{-1}(x)$ to be defined, the arguments of both natural logarithms must be positive, i.e. $x + 2 > 0 \Rightarrow x > -2$ and $-\ln(x + 2) > 0 \Rightarrow \ln(x + 2) < 0 \Rightarrow x + 2 < 1 \Rightarrow x < -1$. So the range is $\{x \in \mathbb{R} | -2 < x < -1\}$.
(Method 3)
Many said that the range is $(-2, -1)$ because the horizontal asymptotes (found in part c) are $y = -2$ and $y = -1$. This receives no credit since this approach usually finds the wrong answer: for example, $f(x) = \frac{e^x+x}{e^x-x}$ has horizontal asymptotes $y = 1$ and $y = -1$, but for all positive x, $f(x) > 1$, so its range is not just $(-1, 1)$. If you really want to mend this argument with horizontal asymptotes, you can say: $h'(x) = \frac{d}{dx}(-e^x)e^{(-e^x)} = -e^x e^{(-e^x)} < 0$ for all x, so $h(x)$ is a decreasing function that is continuous on all of $\mathbb{R}$, so its range is the interval between $\lim_{x\to\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$, i.e. the interval $(-2, -1)$.

(b) (Method 1)
h has no vertical asymptotes because it is continuous on all of $\mathbb{R}$.
(Method 2)
The range of h is $(-2, -1)$, so $|h(x)|$ cannot be arbitrarily large. Hence $\lim_{x\to a^-} h(x)$ and $\lim_{x\to a^+} h(x)$ cannot be ∞ or $-\infty$.

(c) Our computations below are aided by making the substitution $t = -e^x$; notice that $\lim_{x\to\infty} t = -\infty$ and $\lim_{x\to-\infty} t = 0$.

$$\begin{aligned}\lim_{x\to\infty} h(x) &= \lim_{t\to-\infty} e^t - 2\\ &= 0 - 2\\ &= -2\end{aligned}$$

$$\begin{aligned}\lim_{x\to-\infty} h(x) &= \lim_{t\to 0} e^t - 2\\ &= e^0 - 2\\ &= 1 - 2\\ &= -1\end{aligned}$$

So horizontal asymptotes are $y = -2$ and $y = -1$.

(d)

$$\begin{aligned}y &= e^{(-e^x)} - 2\\ y + 2 &= e^{(-e^x)}\\ \ln(y+2) &= -e^x\\ -\ln(y+2) &= e^x\\ \ln(-\ln(y+2)) &= x\end{aligned}$$

So $h^{-1}(x) = \ln(-\ln(x+2))$.

3. (? points) Let

$$f(x) = \begin{cases} ax + b, & x < 1 \\ x^4 + x + 1, & x \geq 1 \end{cases}$$

Find all a and b such that function $f(x)$ is differentiable.

Solution:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (ax + b)$$
$$= a + b$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^4 + x + 1)$$
$$= 3$$

So $f(x)$ is continuous at $x = 1$ if and only if $a + b = 3$.

$$\lim_{x\to 1^-} f'(x) = \lim_{x\to 1^-} a$$
$$= a$$

$$\lim_{x\to 1^+} f'(x) = \lim_{x\to 1^+} (4x^3 + 1)$$
$$= 5$$

So $f'(x)$ is differentiable at $x = 1$ if and only if $a = 5$. So $b = -2$.

4. (? points) Derive the formula $\frac{\mathrm{d}}{\mathrm{d}x} a^x = M(a) a^x$ directly from the definition of the derivative, and identify $M(a)$ as a limit.

Solution:

$$\begin{aligned}
\frac{d}{dx}(a^x) &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} \\
&= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} \\
&= a^x \underbrace{\lim_{h \to 0} \frac{a^h - 1}{h}}_{M(a)} \\
&= M(a) a^x
\end{aligned}$$

End of Quiz 1