Differentiation

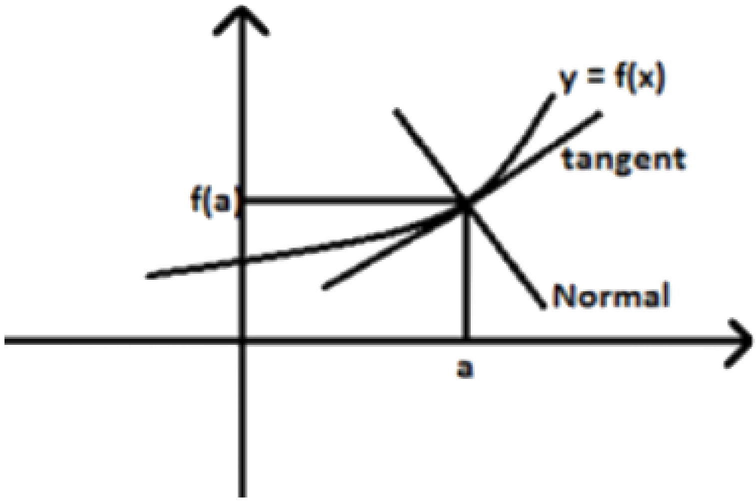
POLYNOMIAL & EXPONENT

1. $\frac{d}{dx}c = 0$, c is a constant
2. $\frac{d}{dx}x^n = n \cdot x^{n-1}$, n is any real number

   Proof. For the case n is a positive integer:

$$\begin{aligned}(a+b)^n &= a^n + C_1^n a^{n-1}b + \cdots + C_n^n b^n \\ \frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0}\frac{\cancel{x^n} + nx^{n-1}h + \cdots + h^n - \cancel{x^n}}{h} \\ &= \lim_{h\to 0} nx^{n-1} + \frac{n(n-1)}{2}x^{n-2} + h^{n-1} \\ &= \quad nx^{n-1}\end{aligned}$$

TANGENT



The equation of the tangent line: $y - f(a) = f'(a)(x - a)$

The equation of the normal line: $y - f(a) = \frac{-1}{f'(a)}(x - a)$

Tangent line L $m_L \cdot m_P = -1 \Rightarrow m_P = \frac{1}{m_L} = \frac{-1}{f'(a)}$

m_P: slope of normal line

See Example 1

3. $\frac{d}{dx} c \cdot f(x) = c \cdot \frac{d}{dx} f(x)$
4. $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$

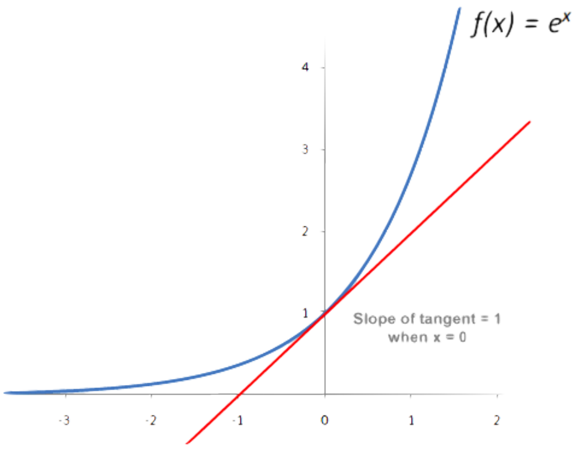
   //NOTE// Where f and g are differentiable function

   See Example 2

5. $\frac{d}{dx} e^x = e^x$

Proof. $$\frac{d}{dx} e^x = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{1} = e^x.$$

Recall the e number is the number such that $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$



See Example 3

PRODUCT & QUOTIENT

💡 $(f \cdot g)' \neq f' \cdot g'$; $\left(\frac{f}{g}\right)' \neq \frac{f'}{g'}$

1. Product rule

   $f \cdot g$: differentiable function

   $\Rightarrow f \cdot g$ is a differentiable function and $(f \cdot g)' = f'g + fg'$

   Proof.

   $$(f \cdot g)' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$
   $$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$
   $$= \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \cdot g(x+h)}_{\text{exists}} + \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \cdot f(x)}_{\text{exists}}$$
   $$= f'(x)g(x) + f(x)g'(x)$$

   //NOTE// Since f and g are differentiable

   See Example 4 & 5

2. Quotient rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

See Example 6 & 7

TRIGONOMETRIC

$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

See Example 8

$$\frac{d^n}{dx^n}\sin x = \begin{cases} \cos x, & n = 4k+1 \\ -\sin x, & n = 4k+2 \\ -\cos x, & n = 4k+3 \\ \sin x, & n = 4k \end{cases}$$

$$\frac{d^n}{dx^n}\cos x = \begin{cases} -\sin x, & n = 4k+1 \\ -\cos x, & n = 4k+2 \\ \sin x, & n = 4k+3 \\ \cos x, & n = 4k \end{cases}$$

See Example 9

CHAIN RULE

$$\frac{d}{dx}(x-1)^2 = \frac{d}{dx}(x^2 - 2x + 1) = 2x - 2$$

$$\frac{d}{dx}(x-1)^{2019} = ?$$

Idea. Set $u = x - 1 \to \frac{du}{dx} = 1$

Known. $\frac{d}{du}u^{2019} = 2019u^{2018}$

$$\frac{d}{du}u^{2019} = \frac{du^{2019}}{du} \cdot \frac{du}{dx} = 2019u^{2018} \cdot 1$$

$$\Rightarrow \frac{d}{dx}(x-1)^{2019} = 2019(x-1)^{2018}$$

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \cdot g$ defined by $F(x) = f(g(x))$ is differentiable at x and $F'(x) = f'(g(x))g'(x)$. If we set $y = f(u)$, $u = g(x)$, $\frac{df(u)}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Given a $F(x)$, we want to find $y = f(u)$ and $u = g(x)$ such that:

1. $F(x) = f(g(x))$
2. $\frac{d}{dx}F(x) = \frac{dy}{du} \cdot \frac{du}{dx}$

   //NOTE// Easier to compute

See Example 10 & 11

POWER RULE

$$\frac{d}{dx}[g(x)]^n = n \cdot [g(x)]^{n-1} \cdot g'(x)$$

See Example 12

$$\frac{d}{dx}b^x = \ln b \cdot b^x, b > 0 \text{ is a constant}$$

Proof. $b^x = (e^{\ln b})^x = e^{\ln b \cdot x}$

$$\frac{d}{dx}b^x = \frac{d}{dx}e^{\ln b \cdot x} = e^{\ln b \cdot x} \cdot \frac{d}{dx}(\ln b \cdot x) = b^x \cdot \ln b$$

See Example 13

INVERSE

$f^{-1}(x)$ is an inverse function of a differentiable function f. Then f^{-1} is also differentiable function.

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. $f(f^{-1}(x)) = x \Rightarrow \frac{d}{dx}f(f^{-1}(x)) = \frac{d}{dx}x = 1$

By chain rule, $f(f^{-1}(x)) \cdot \frac{d}{dx} f^{-1}(x) = 1$

$\therefore \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$.

$$(f^{-1})'(b) = \frac{1}{f'(a)} \text{ where } f(a) = b$$

See Example 14

IMPLICIT

Circle: $x^2 + y^2 = 25 \Rightarrow y = \sqrt{25 - x^2} \text{ or } -\sqrt{25 - x^2}$

//NOTE// y is depend on x

Q. How to compute $\frac{dy}{dx}$?

A. Method of implicit differentiation

Steps:

1. Differentiating both sides of the equation
2. Solving the equation obtain in step 1 for $\frac{dy}{dx}$

See Example 15-17

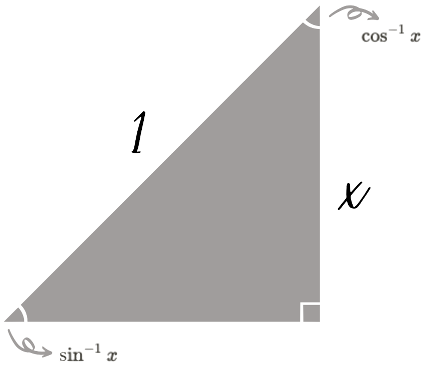
INVERSE

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\csc^{-1} x + \sec^{-1} x = \frac{\pi}{2}$$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$



Proof. Set $y = \sin^{-1} x$, then $\sin y = x$. By implicit differentiation,

$$\frac{d}{dx} \sin y = \frac{d}{dx} x \Rightarrow \cos y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}$$

Since $y = \sin^{-1} x$, we have $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$.

Therefore, $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$.

$$\frac{d}{dx} \cos^{-1} x = \frac{d}{dx} \left(\frac{\pi}{2} - \sin^{-1} x \right) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

$$\frac{d}{dx} \cot^{-1} x = -\frac{1}{1 + x^2}$$

$$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}$$

$$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}$$

See Example 18

LOGARITHMIC

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

Proof. Set $y = \ln x$. We have $e^y = x$ and $\dfrac{d}{dx}e^y = \dfrac{d}{dx}x = 1.$

$$\Rightarrow e^y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Proof. $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$

$f(x) = e^x;\ f^{-1}(x) = \ln x;\ f'(x) = e^x$

$(\ln x)' = \dfrac{1}{f'(\ln x)} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$

$$\frac{d}{dx}\log_a x = \frac{1}{x \ln a},\ a > 0,\ a \neq 1$$

Proof. $\log_a x = \dfrac{\ln x}{\ln a}$

$$\frac{d}{dx}\log_a x = \frac{1}{\ln a} \cdot \frac{d}{dx}\overbrace{\ln x}^{\frac{1}{x}} = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\ln g(x) = \frac{g'(x)}{g(x)} \Rightarrow g'(x) = g(x) \cdot \frac{d}{dx}\ln g(x)$$

//NOTE// By Chain Rule

See Example 19

Given $y = f(x);\ \dfrac{dy}{dx} = ?$

1. $\ln y = \ln f(x)$

   Simplify $\ln f(x)$

2. $y' = y \cdot \dfrac{d}{dx}\ln f(x)$

$\dfrac{d}{dx}\ln g(x) = \dfrac{g'(x)}{g(x)} \Rightarrow g'(x) = g(x) \cdot \dfrac{d}{dx}\ln g(x)$

//NOTE// By Chain Rule

$$\frac{d}{dx}\ln|x| = \frac{1}{x},\ x \neq 0$$

Proof. For $x > 0$, $\dfrac{d}{dx}\ln|x| = \dfrac{d}{dx}\ln x = \dfrac{1}{x}$

For $x < 0$, $\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{(-x)'}{-x} = \frac{-1}{-x} = \frac{1}{x}$

See Example 20

$$\frac{d}{dx}[f(x)]^{g(x)}$$ // use logarithmic differentiation

See Example 21

$$e = \lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{h\to\infty}\left(1+\frac{1}{h}\right)^h$$

Proof. $f(x) = \ln x$; $f'(x) = \frac{1}{x}$; $f'(1) = 1$

$$1 = f'(1) = \lim_{h\to 0}\frac{\ln(1+h) - \ln 1}{h} = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

$$e = e^1 = e^{\lim_{h\to 0}\ln(1+h)^{\frac{1}{h}}} = \underbrace{e^{\ln(\lim_{h\to 0}(1+h)^{\frac{1}{h}})}}_{\ln x \text{ is continuous}} = \lim_{h\to 0}(1+h)^{\frac{1}{h}}$$

Set $h = \frac{1}{n}$; $n \to \infty \Rightarrow h \to 0^+$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{h\to 0^+}(1+h)^{\frac{1}{h}} = e$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

RELATED RATES

Given $y = f(u)$; $u = g(t)$

$\frac{dy}{du}$ = the rate of change of y with respect to u

$\frac{du}{dt}$ = the rate of change of u with respect to t

Q. How to know the rate of change of y with respect to t?

A. By chain rule, $\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt}$.

See Example 22 & 23

LINEAR APPROXIMATION

By definition of the derivative, we have $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$.

By definition of limit, we have $\frac{f(x) - f(a)}{x - a} \approx f'(a)$ when $x \approx a$.

Then the approximation $f(x) \approx f(a) + f'(a)(x - a)$ is called the linear approximation of f at $x = a$.

Def. $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a.

See Example 24

DIFFERENTIALS

$y = f(x)$; f : differentiable function

The differential dy is defined by $dy = f'(x)dx \Rightarrow \frac{dy}{dx} = f'(x)$

Δx : change in x. The corresponding change in y is $\Delta y = f(x + \Delta x) - f(x)$
\\ Change of f from x to $x + \Delta x$

\\ $dx = \Delta x$

See Example 25

CONCLUSION

1. When Δx becomes smaller, the approximation $\Delta y \approx dy$ becomes better.
2. For more complicated function, to estimate the change, the differential is useful.

MARGINAL COST

$C(x)$: cost function

$C(x+1) - C(x) \approx C'(x)$

EXAMPLES

1. Find the tangent line and normal line to the curve $y = x\sqrt{x}$ at $(1, 1)$

   Sol. $y = x^{\frac{3}{2}}$; $\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}}$

   The equation of the tangent line is

   $$\begin{aligned} y - 1 = \frac{3}{2}(1)^{\frac{1}{2}}(x-1) \rightarrow y - 1 &= \frac{3}{2}x - \frac{3}{2} \\ y &= \frac{3}{2}x - \frac{1}{2} \end{aligned}$$

   The equation of the normal line is

   $$\begin{aligned} y - 1 = \frac{-1}{\frac{3}{2}(1)^{\frac{1}{2}}}(x-1) \rightarrow y - 1 &= -\frac{2}{3}x + \frac{2}{3} \\ y &= -\frac{2}{3}x + \frac{5}{3} \end{aligned}$$

2. Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is horizontal

   Sol. $y' = 4x^3 - 12x$

   The tangent line is horizontal means $y' = 0$. To solve $y' = 0$, we have $4x^3 - 12x = 0 \Rightarrow x = 0, \ \pm\sqrt{3}$. So, the points are $(0, 4)$, $(\sqrt{3}, -5)$, $(-\sqrt{3}, -5)$.

3. At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

   Sol. Since the tangent line is parallel to the line $y = 2x$, we have $y' = 2$. So, we need to solve $2 = y' = e^x$. Then we have $x = \ln 2$. Therefore, the point is $(\ln 2, e^{\ln 2}) \rightarrow (\ln 2, 2)$.

4. (a) $f(x) = xe^x$; $f'(x) = ?$

   $u = x \rightarrow u' = 1$

$v = e^x \to v' = e^x$
$f'(x) = e^x + xe^x = e^x(1 + x)$

(b) $f''(x) = ?$

$u = e^x \to u' = e^x$
$v = 1 + x \to v' = 1$
$f''(x) = e^x(1 + x) + e^x = e^x(2 + x)$

Therefore, $f^n(x) = (x + n)e^x$. To prove $f^n(x) = (x + n)e^x$, we use the induction.

For $n = 1$,

$$f'(x) = (x + 1)e^x.$$

Assume $n = k$,

$$f^k(x) = (x + k)e^x.$$

For $n = k + 1$,

$$f^{k+1}(x) = [(x + k)e^x]' = e^x + (x + k)e^x + (x + (k + 1))e^x.$$

By induction, we have $f^n(x) = (n + x)e^x$.

5. $f(x) = \sqrt{x}g(x)$; $g(4) = 2$; $g'(4) = 3$; $f'(4) = ?$

Sol. $f'(x) = (\sqrt{x})'g(x) + \sqrt{x}g'(x)$
$f'(4) = \frac{1}{2}(4)^{-\frac{1}{2}} \cdot 2 + \sqrt{4} \cdot 3 = 6\frac{1}{2}$

6. $y = \dfrac{x^2 + x - 2}{x + 6}$; $y' = ?$

Sol. $u = x^2 + x - 2 \to u' = 2x + 1$
$v = x + 6 \to v' = 1$
$\left(\dfrac{f}{g}\right)' = \dfrac{(2x + 1)(x + 6) - (x^2 + x - 1)}{(x + 6)^2}$

7. $F(x) = \dfrac{3x^2 + 2\sqrt{x}}{x}$; $F'(x) = ?$

Sol. $F(x) = \dfrac{3x^2 + 2\sqrt{x}}{x} = 3x + 2x^{-\frac{1}{2}}$
$F'(x) = 3 + 2\left(-\dfrac{1}{2}\right)x^{-\frac{3}{2}} = 3 - x^{-\frac{3}{2}}$

8. $f(x)=\dfrac{\sec x}{1+\tan x}$, $f(x)=?$ What value of x does the graph of $f(x)$ has a horizontal tangent?

Sol. $f(x)=\dfrac{\frac{1}{\cos x}}{1+\frac{\sin x}{\cos x}}=\dfrac{\frac{1}{\cos x}}{\frac{(\cos x+\sin x)}{\cos x}}=\dfrac{1}{\sin x+\cos x}$

$$f'(x)=\frac{0-(\cos x-\sin x)}{(\sin x+\cos x)^2}=\frac{\sin x-\cos x}{(\sin x+\cos x)^2}$$

The graph of f has a horizontal tangent at x means $f'(x)=0$.

$f'(x)=0\rightarrow -\cos x+\sin x=0\rightarrow \sin x=\cos x\rightarrow \tan x=1\rightarrow$ $x=\dfrac{\pi}{4}+n\pi$, n is any integer.

9. $\dfrac{d^2}{dx^2}x=-x\Rightarrow x=\sin x$

10. $f(x)=\sqrt{x^2+1}$; $f'(x)=?$

Sol. Set $u=x^2+1$; $\dfrac{du}{dx}=2x$; $y=\sqrt{u}$; $\dfrac{dy}{du}=\dfrac{1}{2\sqrt{u}}$

$$f'(x)=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{2\sqrt{u}}\cdot 2x=\frac{x}{\sqrt{x^2+1}}$$

If we set $u=x^2$; $y=\sqrt{u+1}$; $\frac{dy}{du}=?$ ✘

11. $\dfrac{d}{dx}\sin^2 x=?$

Sol. Set $u=\sin x$; $y=u^2$; $\dfrac{du}{dx}=\cos x$; $\dfrac{dy}{du}=2u$

By chain rule, $\dfrac{d}{dx}\sin^2 x=\dfrac{dy}{du}\cdot\dfrac{du}{dx}=2u\cdot\cos x=2\sin x\cos x=$ $\sin 2x$

12. $f(x)=\dfrac{1}{\sqrt[3]{x^2+x+1}}$; $f'(x)=?$

Sol. $f'(x)=-\dfrac{1}{3}(x^2+x+1)^{-\frac{4}{3}}\cdot\dfrac{d}{dx}(x^2+x+1)=-\dfrac{1}{3}(x^2+x+1)^{-\frac{4}{3}}(2x+1)$

13. $f(x)=\sin(\cos(\tan x))$; $f'(x)=?$

Sol. $f'(x)=\cos(\cos(\tan x))\cdot\dfrac{d}{dx}\cos(\tan x)$

$$=\cos(\cos(\tan x))\cdot(-\sin(\tan x))\cdot\frac{d}{dx}\tan x$$

$$= -\cos(\cos(\tan x)) \cdot \sin(\tan x) \cdot \sec^2 x$$

14. $f(4) = 5;\ f'(4) = \dfrac{2}{3};\ (f^{-1})'(5) = ?$

Sol. $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))}$; since $f(4) = 5$, $f^{-1}(5) = 4$

$$(f^{-1})'(5) = \frac{1}{f'(4)} = \frac{3}{2}$$

15. (a) $x^2 + y^2 = 25;\ \dfrac{dy}{dx} = ?$

Sol. $\dfrac{d}{dx}(x^2 + y^2) = \dfrac{d}{dx}25 \Rightarrow \dfrac{d}{dx}x^2 + \dfrac{d}{dx}\underbrace{y^2}_{y=f(x)} = 0$

By chain rule, $2x + 2y\dfrac{dy}{dx} = 0 \Rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$

(b) The equation of the tangent to the circle $x^2 + y^2 = 25$ at $(3, 4)$

Sol. The slope of the equation tangent is

$$\left.\frac{dy}{dx}\right|_{(x,y)=(3,4)} = -\frac{3}{4}$$

The equation of the tangent is $(y - 4) = -\dfrac{3}{4}(x - 3)$.

16. y' if $\sin(x + y) = y^2 \cos x$

Sol. $\dfrac{d}{dx}(\sin(x + y)) = \dfrac{d}{dx}(y^2 \cos x)$

$$\cos(x + y) \cdot \frac{d}{dx}(x + y) = \left(\frac{d}{dx}y^2\right)\cos x + y^2\left(\frac{d}{dx}\cos x\right)$$

$$\cos(x + y)\left(1 + \frac{dy}{dx}\right) = 2y\frac{dy}{dx}\cos x + (-\sin x)y^2$$

$$\cos(x + y) + \cos(x + y)\frac{dy}{dx} = 2y\cos x\frac{dy}{dx} - y^2 sinx$$

$$(\cos(x + y) - 2y\cos x)\frac{dy}{dx} = -y^2 \sin x - \cos(x + y)$$

$$\frac{dy}{dx} = \frac{y^2 \sin x + \cos(x + y)}{2y\cos x - \cos(x + y)}$$

17. y'' if $x^4 + y^4 = 16$

Sol. $\dfrac{d}{dx}(x^4+y^4)=\dfrac{d}{dx}16 \Rightarrow 4x^3+4y^3\dfrac{dy}{dx}=0 \Rightarrow \dfrac{dy}{dx}=-\dfrac{x^3}{y^3}$

$$y''=(y')'=\left(-\frac{x^3}{y^3}\right)'=\frac{-3x^2y^3+x^3\cdot 3y^2\frac{dy}{dx}}{y^6}$$

$$=\frac{-3x^2y^3+x^3\cdot 3y^2\left(-\frac{x^3}{y^3}\right)}{y^6}=\frac{-3x^2y^3-3x^6y^{-1}}{y^6}$$

$$=\frac{-3x^2(\overbrace{y^4+x^4}^{16})}{y^7}=-48x^2y^{-7}$$

18. $f(x)=x\tan^{-1}\sqrt{x};\ f'(x)=?$

Sol. $f'(x)=x'\tan^{-1}\sqrt{x}+x(\tan^{-1}\sqrt{x})'$

$$=\tan^{-1}\sqrt{x}+x\cdot\frac{1}{2}x^{-\frac{1}{2}}\cdot\frac{1}{1+x}$$

$$=\tan^{-1}\sqrt{x}+\frac{\sqrt{x}}{2(1+x)}$$

19. $\dfrac{d}{dx}\ln\left(\dfrac{x+1}{\sqrt{x-2}}\right)$

Sol. $\dfrac{d}{dx}\ln\underbrace{\left(\dfrac{x+1}{\sqrt{x-2}}\right)}_{g(x)}=\dfrac{1}{\frac{x+1}{\sqrt{x-2}}}\left(\dfrac{x+1}{\sqrt{x-2}}\right)'=\dfrac{\sqrt{x-2}}{x+1}\cdot$

$$\frac{\sqrt{x-2}-(x+1)\cdot\frac{1}{2\sqrt{x-2}}}{(\sqrt{x-2})^2}$$

$$=\frac{(x-2)-\frac{x+1}{2}}{(x+1)(x-2)}=\frac{x-5}{2(x+1)(x-2)}$$

Sol. $\ln\dfrac{x+1}{\sqrt{x-2}}=\ln(x+1)-\dfrac{1}{2}\ln(x-2)$

$$\frac{d}{dx}\ln\frac{x+1}{\sqrt{x-2}}=\frac{d}{dx}\ln(x+1)-\frac{1}{2}\cdot\frac{d}{dx}\ln(x-2)$$

$$=\frac{1}{x+1}-\frac{1}{2(x-2)}=\frac{x-5}{2(x+1)(x-2)}$$

20. $y=\dfrac{x^{\frac{4}{3}}\sqrt{x^2+1}}{(3x+2)^5};\ y'=?$

Sol. $\ln y=\ln\dfrac{x^{\frac{4}{3}}\sqrt{x^2+1}}{(3x+2)^5}=\dfrac{4}{3}\ln x+\dfrac{1}{2}\ln(x^2+1)-5\ln(3x+2)$

$$\frac{d}{dx}\ln y = \frac{4}{3}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{2x}{x^2+1} - 5\cdot\frac{3}{3x+2}$$
$$\frac{dy}{dx} = y\left(\frac{4}{3x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right) =$$
$$\frac{x^{\frac{4}{3}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{4}{3x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

21. $y = x^{\sqrt{x}}; y' = ?$

Sol. $\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x$

$$\Rightarrow \frac{d}{dx}\ln y = \frac{d}{dx}\sqrt{x}\ln x$$

$$\frac{dy}{dx}\cdot\frac{1}{y} = \frac{1}{2\sqrt{x}}\ln x + \frac{\sqrt{x}}{x}$$

$$\frac{dy}{dx} = y\left(\frac{1}{\sqrt{x}}\left(\frac{\ln x}{2} + 1\right)\right) = \frac{x^{\sqrt{x}}}{\sqrt{x}}\left(\frac{\ln x}{2} + 1\right)$$

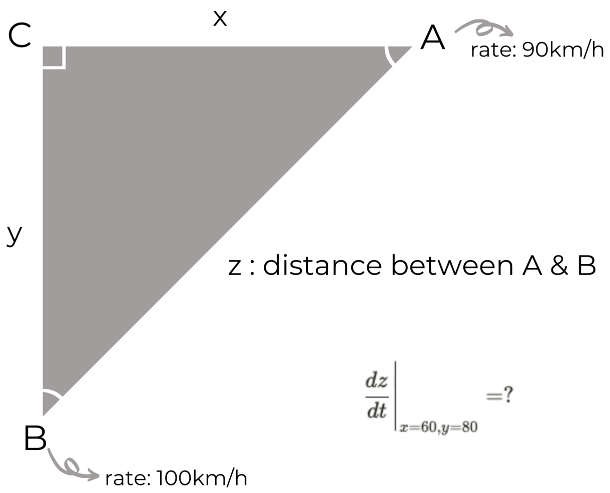
22. There is a balloon is a balloon.

Volume $\uparrow 100cm^3/s$

radius $\uparrow ?cm/s$ when diameter = $50cm$

Sol. Set V is the volume, r is the radius. Then $V(r) = \frac{4}{3}\pi r^3$. By assumption, $\dfrac{dV}{dt} = 100$ and $r = 25$. We want to know $\left.\dfrac{dr}{dt}\right|_{r=25}$.
Differentiating $V(r)$ on both sides with the respect to t, we have

$$\frac{dV}{dt} = 4\pi r^2\cdot\frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\cdot\frac{dV}{dt}$$
$$\left.\frac{dr}{dt}\right|_{r=25} \Rightarrow \frac{dr}{dt} = \frac{1}{4\pi\cdot 25^2}\cdot 100 = \frac{1}{25\pi}cm/s$$



23.

Sol. Set x is the distance between A and C, y is the distance between B and C where C is the intersection. Let z is the distance between A and B.

$\frac{dx}{dt} = -90; \frac{dx}{dt} = -100$

Also, $z^2 = x^2 + y^2$

$\frac{d}{dt}(z^2) = \frac{d}{dt}(x^2 + y^2)$

$\not{2}z\frac{dz}{dt} = \not{2}x\frac{dx}{dt} + \not{2}\frac{dy}{dt}$

When $x = 60$, $y = 80$, we have $z = 100$ and $\frac{dz}{dt} = -134km/h.$

24. (a) Find the linearization of $f(x) = \sin^{-1} x$ at $x = 0.5$

Sol. $f(x) = \sin^{-1} x$; $f'(x) = \frac{1}{\sqrt{1-x^2}}$

The linearization of f at $x = 0.5$ is $f(0.5) + f'(0.5)(x - 0.5)$

$$L(x) = \frac{\pi}{6} + \frac{1}{\sqrt{1-\frac{1}{4}}}\left(x - \frac{1}{2}\right) = \frac{\pi}{6} + \frac{2}{\sqrt{3}}\left(x - \frac{1}{2}\right)$$

(b) Use linear approximation to estimate $\sin^{-1} 0.49$

Sol. $\sin^{-1} 0.49 \approx L(0.49) = f(0.5) + f'(0.5)(0.49 - 0.5)$

$$= \frac{\pi}{6} + \frac{2}{\sqrt{3}}(-0.01) = \frac{\pi}{6} - \frac{2}{100\sqrt{3}}$$

25. Compare Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes:

(a) 2 to 2.05

Sol. $\Delta y = f(2.05) - f(2) = 0.717625$
$dy = f'(x)dx = (3x^2 + 2x - 2)dx$
From 2 to 2.05, $dx = 0.05 \to dy = f'(2) \cdot 0.05 = 14 \cdot 0.05 = 0.7$

(b) 2 to 2.01

Sol. $\Delta y = f(2.01) - f(2) = 0.140701$
$dy = f'(2)(2.01 - 2) = 14 \cdot 0.01 = 0.14$