1. (17 pts) Evaluate the following limits or show that they do not exist.

   (a) $\lim_{x\to-\infty}\left(\sqrt{x^2+x+1}+x\right)$.

   (b) $\lim_{x\to0}(1-\cos x)\cdot\left[\!\left[\frac{1}{x^2}\right]\!\right]$.(Here $[\![x]\!]$ denotes the 'greatest integer function' and satisfies $x-1<[\![x]\!]\le x$.)

   (c) $\lim_{x\to0}\frac{\sin x}{\sqrt{1-\cos(2x)}}$.

   (d) $\lim_{x\to0}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}$.

Solution:

(a)

Marking scheme :
1M - Attempt to rationalise
1M - Rationalise correctly
1M - Divide both the numerator & denominator by x (correctly)
1M - Correct numerical answer

Remark : If a candidate attempts to use L'Hospital's rule after rationalisation, the last 2M will only be awarded if the last displayed numerical answer is correct.

(Sample solution)

$$\lim_{x\to-\infty}(\sqrt{x^2+x+1}+x)=\underbrace{\lim_{x\to-\infty}\frac{x+1}{\sqrt{x^2+x+1}-x}}_{(2M)}=\underbrace{\lim_{x\to-\infty}\frac{1+\frac{1}{x}}{-\sqrt{1+\frac{1}{x}+\frac{1}{x}}-1}}_{(1M)}=\underbrace{-\frac{1}{2}}_{(1M)}.$$

(b)

Marking scheme :
2M - Correct bounds of the Gaussian/floor function
1M - Correct evaluation of the limit $\lim_{x\to0}\frac{1-\cos x}{x^2}$
1M - Use of squeeze theorem

(Sample solution)

$$\underbrace{\frac{1}{x^2}-1<\left[\!\left[\frac{1}{x^2}\right]\!\right]\le\frac{1}{x^2}}_{(2M)}$$

$$\Longrightarrow\frac{1-\cos x}{x^2}-(1-\cos x)<(1-\cos x)\cdot\left[\!\left[\frac{1}{x^2}\right]\!\right]\le\frac{1-\cos x}{x^2}$$

$$\text{Since }\lim_{x\to0}\frac{1-\cos x}{x^2}=\lim_{x\to0}\frac{1-\cos x}{x^2}-(1-\cos x)=\underbrace{\frac{1}{2}}_{(1M)},$$

the Squeeze Theorem implies that $\lim_{x\to0}(1-\cos x)\cdot\left[\!\left[\frac{1}{x^2}\right]\!\right]=\frac{1}{2}$. $(1M)$

(c)

Marking scheme :
1M - use of double angle formula or the identity $1-\cos^2(2x) = \sin^2(2x)$ (appropriately)
1M - consider the left and right limits
1M - correct evaluation of one of these limits
1M - correct conclusion

Remark.
1) At most 1M will be given (very generously) to candidates who attempt with L'Hopital's rule.
2) At most 2M will be awarded to candidates who have only evaluated one-side of the limit.

(Sample solution 1)

$$\text{Since } \frac{\sin x}{\sqrt{1-\cos(2x)}} = \underbrace{\frac{\sin x}{\sqrt{1-(1-2\sin^2 x)}}}_{(1M)} = \frac{\sin x}{\sqrt{2}\cdot|\sin x|} \text{ and}$$

$$\underbrace{\begin{cases} \lim_{x\to 0^+} \frac{\sin x}{\sqrt{2}\cdot|\sin x|} & = \underbrace{\frac{1}{\sqrt{2}}}_{(1M)} \\ \lim_{x\to 0^-} \frac{\sin x}{\sqrt{2}\cdot|\sin x|} & = -\frac{1}{\sqrt{2}} \end{cases}}_{(1M)},$$

we conclude that the limit does not exist (1M).

(Sample solution 2)

$$\text{Since } \frac{\sin x}{\sqrt{1-\cos(2x)}} = \underbrace{\frac{\sin x\cdot\sqrt{1+\cos 2x}}{\sqrt{\sin^2(2x)}}}_{(1M)} = \frac{\sin x\cdot\sqrt{1+\cos 2x}}{|\sin 2x|} \text{ and}$$

$$\underbrace{\begin{cases} \lim_{x\to 0^+} \frac{\sin x\cdot\sqrt{1+\cos 2x}}{|\sin 2x|} & = \lim_{x\to 0^+} \frac{\sin x}{x}\cdot\frac{2x}{\sin 2x}\cdot\frac{\sqrt{1+\cos 2x}}{2} = \underbrace{\frac{\sqrt{2}}{2}}_{(1M)}, \\ \lim_{x\to 0^-} \frac{\sin x\cdot\sqrt{1+\cos 2x}}{|\sin 2x|} & = -\frac{\sqrt{2}}{2} \end{cases}}_{(1M)}$$

we conclude that the limit does not exist (1M).

(d)

Marking scheme :
1M - for taking logarithms of the given expression appropriately
1M - for tidying up the expression after taking log (correctly)
2M - for using L-H's rule and mentioning the correct indeterminate form
1M - for the correct numerical answer

Remark.
1) 1M will be taken away for forgetting to make reference to the appropriate indeterminate form
2) 1M will be taken away for candidates who leave $-\frac{1}{2}$ as the 'final answer'.

(Sample solution 1)

Let $y = \left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}}$. Then $\underbrace{\ln y = \frac{1}{x} \cdot \ln\left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)}_{(1M)} = \frac{\frac{\ln(1+x)}{x} - \ln e}{x} = \underbrace{\frac{\ln(1+x) - x}{x^2}}_{(1M)}$. Therefore,

$$\ln\left(\lim_{x\to 0} y\right) = \lim_{x\to 0} \ln y = \underbrace{\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{2x}}_{(2M)} = \lim_{x\to 0} \frac{-1}{2(1+x)} = -\frac{1}{2}.$$

Hence, $\lim_{x\to 0} y = \underbrace{e^{-\frac{1}{2}}}_{(1M)}$.

(Sample solution 2)

$$\lim_{x\to 0}\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]^{\frac{1}{x}} = \underbrace{\lim_{x\to 0} e^{\frac{\ln\left[\frac{(1+x)^{\frac{1}{x}}}{e}\right]}{x}}}_{(1M)} = \underbrace{e^{\lim_{x\to 0} \frac{\ln(1+x) - x}{x^2}}}_{(1M)} \underbrace{\overset{\frac{0}{0}}{=} e^{\lim_{x\to 0} \frac{\frac{1}{1+x} - 1}{2x}}}_{(2M)} = e^{\lim_{x\to 0} \frac{-1}{2(1+x)}} = \underbrace{e^{\frac{-1}{2}}}_{(1M)}.$$

2. (12 pts)

(a) $f(x) = e^{x^2 \cdot \sec x}$. Find $f'(x)$.

(b) $f(x) = x \cdot \cos^{-1} x - \sqrt{1-x^2}$. Find $f'(x)$.

(c) $f(x) = (1 + \sin x)^{\cot x}$. Find $f'(x)$.

Solution:

(a)

$$f'(x) = e^{x^2 \cdot \sec x} \cdot \frac{d}{dx}(x^2 \cdot \sec x) \text{ (2 pts)} = e^{x^2 \cdot \sec x} \cdot (2x \cdot \sec x + x^2 \cdot \sec x \cdot \cdot \tan x). \text{ (2 pts)}$$

(b)

$$\begin{aligned} f'(x) &= \cos^{-1} x + x \cdot \frac{-1}{\sqrt{1-x^2}} - \frac{d}{dx}\sqrt{1-x^2} \text{ (2 pts)} \\ &= \cos^{-1} x + x \cdot \frac{-1}{\sqrt{1-x^2}} - \frac{-1}{\sqrt{1-x^2}} = \cos^{-1} x. \text{ (2 pts)} \end{aligned}$$

(c) (Method 1)

$$\begin{aligned} f(x) &= (1 + \sin x)^{\cot x} = e^{\cot x \cdot \ln(1+\sin x)}; \text{ (1 pt)} \\ f'(x) &= e^{\cot x \cdot \ln(1+\sin x)} \cdot \frac{d}{dx}[\cot x \cdot \ln(1 + \sin x)] \text{ (2 pts)} \\ &= (1 + \sin x)^{\cot x} \cdot [-\csc^2 x \cdot \ln(1 + \sin x) + \cot x \cdot \frac{\cos x}{1 + \sin x}]. \text{ (1 pt)} \end{aligned}$$

(Method 2)

$$\begin{aligned} f(x) &= (1 + \sin x)^{\cot x} \\ \ln f(x) &= \ln(1 + \sin x)^{\cot x} = \cot x \cdot \ln(1 + \sin x) \text{ (1 pt)} \\ \Rightarrow \frac{f'(x)}{f(x)} &= -\csc^2 x \cdot \ln(1 + \sin x) + \cot x \cdot \frac{\cos x}{1 + \sin x} \text{ (2 pts)} \\ \Rightarrow f'(x) &= f(x) \cdot [-\csc^2 x \cdot \ln(1 + \sin x) + \cot x \cdot \frac{\cos x}{1 + \sin x}] \\ &= (1 + \sin x)^{\cot x} \cdot [-\csc^2 x \cdot \ln(1 + \sin x) + \cot x \cdot \frac{\cos x}{1 + \sin x}] \text{ (1 pt)} \end{aligned}$$

3. (15 pts) Consider the function

$$f(x) = \begin{cases} \sin(3x) + \cos(2x) + A & ,\text{if } x \le 0 \\ x^2 \cdot \ln x & ,\text{if } x > 0 \end{cases}$$

where A is an auxiliary constant. It is given that f is continuous everywhere.

(a) (4 pts) Find the value of A.

(b) (5 pts) Is f differentiable at $x = 0$? Explain.

(c) (6 pts) Prove that for $x > 1$, the equation $f(x) = e$ has a unique solution.

Solution:

(a)

Marking scheme :
1M - Correct definition of continuity
2M - Correct derivation and evaluation of $\lim\limits_{x\to 0^+} x^2 \ln x$
1M - Correct value of A

Remark : 1M is taken off if a candidate writes $\lim\limits_{x\to 0^+} x^2 \ln x = 0$ without any explanations.

Sample Solution of (a).
By continuity, we have $\underbrace{\lim_{x\to 0^+} f(x) = f(0)}_{(1M)}$. Since

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 \ln x = \underbrace{\lim_{x\to 0^+} \frac{\ln x}{1/x^2} = \lim_{x\to 0^+} \frac{1/x}{-2/x^3}}_{(1M)} = \underbrace{0}_{(1M)},$$

we have $\underbrace{A = -1}_{(1M)}$.

(b)

Marking scheme :
1M - Correct definition of differentiability
1M - Correct evaluation of $\lim\limits_{h\to 0^+} \frac{f(h)-f(0)}{h}$
2M - Correct derivation and evaluation of $\lim\limits_{h\to 0^-} \frac{f(h)-f(0)}{h}$
1M - Correct conclusion

Remark :
1. At most 1M will be (generously) awarded to candidates who proved the irrelevant fact that $\lim\limits_{x\to 0^+} f'(x) \neq \lim\limits_{x\to 0^-} f'(x)$.

2. In this part, if candidates write $\lim\limits_{x\to 0^+} x \ln x = 0$ without any explanations, no deductions will be made.

3. Candidates will lose 1M for the evaluation of $\lim\limits_{h\to 0^-} \frac{f(h)-f(0)}{h}$ if their answers to (a) is incorrect. (So they can earn at most 4M in this part)

4. Candidates may get 2M or 3M if they have only considered one of the one-sided limits of the difference quotient, depending on which side has been computed.

Sample Solution of (b).

$$\underbrace{\lim_{h\to0^+}\frac{f(h)-f(0)}{h}}_{(1M)} = \lim_{h\to0^+} h\ln h = \lim_{h\to0^+}\frac{\ln h}{1/h} = \lim_{h\to0^+}\frac{1/h}{-1/h^2} = \underbrace{0}_{(1M)}$$

$$\lim_{h\to0^-}\frac{f(h)-f(0)}{h} = \underbrace{\lim_{h\to0^-}\frac{\sin(3h)+\cos(3h)-1}{h} = \lim_{h\to0^-} 3\cdot\frac{\sin(3h)}{3h}+\frac{\cos(3h)-1}{h}}_{(1M)} = \underbrace{3}_{(1M)}$$

Therefore, the limit $\lim_{h\to0}\frac{f(h)-f(0)}{h}$ does not exist and hence f is not differentiable at $x=0$ (1M).

(c)

> Marking scheme :
>
> <u>Existence</u>
> 1M - Mentioning f (or certain piece of of f) is continuous
> 1M - Evaluating f at two (appropriate) values
> 1M - Conclusion (it doesn't matter if 'IVT' is explicitly mentioned or not)
> <u>Uniqueness</u>
> 1M - computing the derivative of f (correctly)
> 1M - mentioning that $f'\neq0$ or $f'>0$
> 1M - referencing MVT/Rolle/'f is (strictly) increasing' to complete the proof
>
> Remark : 1M will be taken off if a student only writes $f'\geq0$.

Sample Solution of (c).

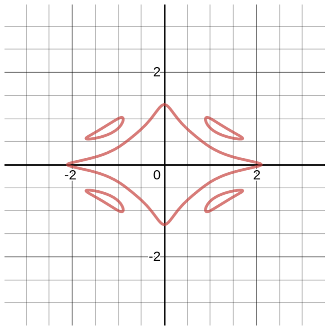
<u>Existence</u>
Since f is continuous (1M)
and $f(1)=0$ and $f(e)=e^2$, (1M)
the Intermediate Value Theorem implies that $\exists c\in(1,e)$ such that $f(c)=e$. (1M)

<u>Uniqueness</u>
Moreover, $f'(x)=x(2\ln x+1)$ (1M)
so $f'(x)>0$ for $x\geq1$ (1M)
and hence this solution is unique as a consequence of the Rolle's Theorem/M.V.T. (1M)
OR *this solution is unique because f is strictly increasing for $x\geq1$.* *(1M)*

<u>Alternative proof for uniqueness</u>
Suppose α and β are two distinct solutions to the equation $f(x)=e$.
Then Rolle's Theorem implies that $f'(c)=0$ for some c between α and β. (1M)
However, $f'(x)=x(2\ln x+1)$ (1M)
can never be zero/ is strictly positive for every $x\geq1$. (1M)

4. (10 pts) The graph given below is defined by the equation

$$\ln(x^2+1)+y^2-\big(\cos(\pi xy)\big)^2-\ln 2=0.$$



(a) (6 pts) Find $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,1)}$.

(b) (4 pts) Let $f(x)$ be the function such that the above equation can be implicitly written as $y=f(x)$ near $(1,1)$. Use the linear approximation of $f(x)$ at $x=1$ to approximate $f(0.98)$.

Solution:

(a) Differentiation of $F(x,y)$ with respect to x gives us

$$\frac{2x}{x^2+1}+2yy'=2\cos(\pi xy)(-\sin(\pi xy))(\pi(1+xy')).$$

Plugging $(x,y)=(1,1)$ into the equation, we obtain $1+2y'=0$, so $y'(1)=\frac{-1}{2}$. Thus, the equation of the tangent line is

$$y=L_1(x)=1-\frac{1}{2}(x-1).$$

Grading Schemes. *:*
1pt for derivative of $\ln x$
1pt for derivative of $\cos x$
3pt for chain rule (loss 3 points if fail twice)
1pt for the equation of tangent line.

(b) The linear approximation, being the tangent line at $x=1$, of $f(0.98)$ is $L_1(0.98)=1.01$.
Grading Schemes. *:*
2 pt for knowing the tangent line is the linear approximation
2 pt for computation correctness

5. (8 pts) Assume that $f(x)$ is continuous on [0,1] and differentiable on (0,1) with $f(0) = 1, f(1) = 0$. Show that there exists $c \in (0,1)$ such that

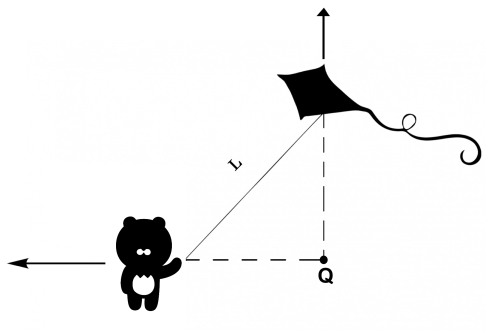
$$f'(c) = -\frac{f(c)}{c}.$$

(Hint: Consider $F(x) = x \cdot f(x),\ x \in [0,1]$.)

Solution:

Consider the function $F(x) = x \cdot f(x)$ defined on $[0,1]$. Then $F(x)$ is continuous on [0,1] and differentiable on (0,1) with $F(0) = F(1) = 0$. (3 pts) By Rolle's theorem(or MVT), there exists $c \in (0,1)$ such that $F'(c) = f(c) + cf'(c) = 0$.(3 pts) That is, $f'(c) = -\frac{f(c)}{c}$ for some $c \in (0,1)$.(2 pts)

6. (12 pts) Ryan is flying a kite with an extendible string of length L (see figure). While the kite is rising vertically above the point Q at a rate of 0.4 m/sec, Ryan is running away from Q at a constant rate of 2 m/sec. At the moment when the kite is 3 m above the ground, the length of the string between Ryan and the kite is 5 m. Find the rate of change of the length of the string at this moment.



Solution:

Set x is the distance between the Ryan and the point Q and y is the distance between the point Q and the kite. (2%)

Then the relation between x, y and L is given by

$$L^2 = x^2 + y^2. \quad (2\%) \tag{1}$$

Differentiating (1) on both sides with respect to t, we have

$$2L\frac{dL}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}. \quad (3\%) \tag{2}$$

When $L = 5$ and $y = 3$, we have $x = \sqrt{5^2 - 3^2} = 4$ (1%).

Since Ryan is running away from Q at a constant rate of 2 m/sec and the kite is rising vertically above the point Q at a rate of 0.4 m/sec, we have

$$\frac{dx}{dt} = 2, \quad \frac{dy}{dt} = 0.4. \quad (2\%)$$

Substituting $x = 4$, $y = 3$, $L = 5$, $\frac{dx}{dt} = 2$ and $\frac{dy}{dt} = 0.4$ into (2), we have

$$2(5)\frac{dL}{dt} = 2(4)(2) + 2(3)(0.4) \quad \Rightarrow \quad \frac{dL}{dt} = \frac{2(4)(2) + 2(3)(0.4)}{10} = 1.84$$

Thus, the rate of change of the length of the string is 1.84 (m/s). (2%)

7. (10 pts) On the xy-plane, a line segment passes through three points $P(x,0)$, $Q(1,a)$ and $R(0,y)$, with $y > a > 0$, $x > 1$. Fix Q, and use the first or second derivative test to show that the length of the segment $\overline{PR}$ can be minimized by adjusting the locations of P and R.

Solution:

Must show a)the problem formulation, b)differentiation to find the criticals, and c)1st/2nd derivative test

(Method 1)

a) Let s denote the slope of any line segment thus formed. Then $s = \frac{y-a}{0-1} = \frac{a-0}{1-x} \Rightarrow y = a + \frac{a}{x-1} = \frac{a\,x}{x-1} > a$ for $x > 1$.

Let ℓ denote the segment's length, then
$\ell(x) = \sqrt{x^2+y^2} = \frac{x}{x-1}\sqrt{(x-1)^2+a^2}$ for $x > 1$ $\cdots$ (4%)

b) $\Rightarrow \ell'(x) = \frac{-\sqrt{(x-1)^2+a^2}}{(x-1)^2} + \frac{x}{\sqrt{(x-1)^2+a^2}} = \frac{(x-1)^3-a^2}{(x-1)^2\sqrt{(x-1)^2+a^2}}$

$= 0$ at $x = x^* = 1 + \sqrt[3]{a^2}$ $\cdots$ (3%)

c) Since $\ell'(x) < 0$ for $x < x^*$, and $\ell'(x) > 0$ for $x > x^*$, a minimum length is confirmed by first derivative test. $\cdots$ (3%)

(Method 2)

a) Let θ denote the angle contained by the segment and the x-axis. Then the segment's length is given by $g(\theta) = \frac{a}{\sin\theta} + \frac{1}{\cos\theta} = \sec\theta + a\csc\theta$. $0 < \theta < \pi/2$ $\cdots$ (4%)

b) Therefore $g'(\theta) = \tan\theta\sec\theta - a\cot\theta\csc\theta = \tan\theta\sec\theta(1 - a\cot^3\theta)$
which shows that $g' = 0$ at $\theta^* = \cot^{-1}\frac{1}{\sqrt[3]{a}}$ $\cdots$ (3%)

c.1) Since $g' < 0$ for $\theta < \theta^*$, $g' > 0$ for $\theta > \theta^*$, a minimum length is confirmed by first derivative test. $\cdots$ (3%)

c.2) Or, $g'' = \sec\theta\,(\tan^2\theta + \sec^2\theta)$ $+a\csc\theta\,(\cot^2\theta + \csc^2\theta) > 0$, for any $0 < \theta < \pi/2$. Hence, a minimum length is confirmed by second derivative test. $\cdots$ (3%)

(Method 3)

Let $m < 0$ be the slope passing through $Q(1,a)$. Then we have $\overleftrightarrow{PR} : y - a = m(x-1)$. That is, $\overleftrightarrow{PR} : y = mx + a - m$ which implies $P(\frac{m-a}{m}, 0)$ and $R(0, a-m)$. (3 pts) Consider the function

$$f(m) := \overline{PR}^2 = \frac{(m-a)^2}{m^2} + (a-m)^2 = (m-a)^2 \cdot [1 + \frac{1}{m^2}], \quad m < 0. \quad \text{(3 pts)}$$

Then

$$f'(m) = 2(m-a)\cdot[1+\frac{1}{m^2}] + (m-a)^2\cdot(\frac{-2}{m^3}) = (m-a)\cdot[2 + \frac{2}{m^2} + (m-a)\cdot\frac{-2}{m^3}]$$
$$= (m-a)\cdot[2 + \frac{2}{m^3}] = 2(m-a)\cdot\frac{m^3+a}{m^3}, \quad m < 0. \quad \text{(2 pts)}$$

The critical number of $f(m)$ is $m = \sqrt[3]{-a} = -\sqrt[3]{a} = -a^{\frac{1}{3}}$. (Note that $m < 0$.) So we have $f'(m) < 0$ on $(-\infty, -a^{\frac{1}{3}})$ and $f'(m) > 0$ on $(-a^{\frac{1}{3}}, 0)$. By the first derivative test,

$$f(-a^{\frac{1}{3}}) = (-a^{\frac{1}{3}} - a)^2 \cdot (1 + \frac{1}{(-a^{\frac{1}{3}})^2}) = [a^{\frac{1}{3}}(1+a^{\frac{2}{3}})]^2 \cdot (1 + a^{\frac{-2}{3}}) = (1+a^{\frac{2}{3}})^2 \cdot (1+a^{\frac{2}{3}}) = (1+a^{\frac{2}{3}})^3$$

is the absolute minimum value of $f(m)$. (2 pts)
That is, $\text{Min}(\overline{PR}) = \sqrt{f(-a^{\frac{1}{3}})} = (1 + a^{\frac{2}{3}})^{\frac{3}{2}}$.

8. (16 pts) Consider the function

$$f(x) = \frac{x(x-8)}{\sqrt{x^2-4}}, \quad |x| > 2 \quad \text{with} \quad f''(x) = \frac{4(x^2-24x+8)}{(x^2-4)^{\frac{5}{2}}}.$$

Fill in each blank below. Show your work (computations and reasoning) in the space following. Put None in the blank if the item asked does not exist.

(a) (6 pts) The asymptotes of $f(x)$ are: $x = \pm 2,\ y = -x+8,\ y = x-8$.

(b) (5 pts) $f(x)$ is increasing on the interval(s): $(-4,-2) \cup (2,\infty)$.

$f(x)$ is decreasing on the interval(s): $(-\infty,-4)$.

Local maximum point(s) of $f(x)$: $(x;y) =$ None (No local maximum).

Local minimum point(s) of $f(x)$: $(x;y) = (-4, 8\sqrt{3})$.

(c) (3 pts) $f(x)$ is concave upward on the interval(s): $(-\infty,-2) \cup (12+2\sqrt{34},\infty)$.

$f(x)$ is concave downward on the interval(s): $(2, 12+2\sqrt{34})$.

The inflection point(s) would occur at $x = 12+2\sqrt{34}$.

(d) (2 pts) Sketch the graph of $y = f(x)$. Indicate, if any, asymptotes, intervals of increase or decrease, concavity, local extreme values, and points of inflection.

Solution:

(a) Since $\lim_{x\to\pm\infty} f(x) = \infty$, there is no horizontal or vertical asymptote of $f(x)$. It is easy to see that the vertical asysmptotes are $x = \pm 2$. (2 pts) Now, we determine the slant asymptote of the $f(x)$. We firstly compute the limits

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{x(x-8)}{x\sqrt{x^2-4}} = \lim_{x\to\infty}\frac{x-8}{\sqrt{x^2}\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{x-8}{|x|\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to\infty}\frac{x-8}{x\sqrt{1-\frac{1}{x^2}}} = 1; \ \text{(1 pt)}$$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{x-8}{\sqrt{x^2}\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to-\infty}\frac{x-8}{|x|\sqrt{1-\frac{1}{x^2}}} = \lim_{x\to-\infty}\frac{x-8}{-x\sqrt{1-\frac{1}{x^2}}} = -1. \ \text{(1 pt)}$$

Then we compute th limits

$$\lim_{x\to\infty}[f(x)-x] = \lim_{x\to\infty}[\frac{x(x-8)}{\sqrt{x^2-4}} - x] = \lim_{x\to\infty}\frac{x[(x-8)-\sqrt{x^2-4}]}{\sqrt{x^2-4}}$$

$$= \lim_{x\to\infty}\frac{x[(x-8)^2-(x^2-4)]}{\sqrt{x^2-4}\cdot[(x-8)+\sqrt{x^2+4}]} = \lim_{x\to\infty}\frac{x(-16x+68)}{\sqrt{x^2-4}\cdot[(x-8)+\sqrt{x^2+4}]} = \frac{-16}{2} = -8; \ \text{(1 pt)}$$

$$\lim_{x\to-\infty}[f(x)+x] = \lim_{x\to-\infty}[\frac{x(x-8)}{\sqrt{x^2-4}} + x] = \lim_{x\to-\infty}\frac{x[(x-8)+\sqrt{x^2-4}]}{\sqrt{x^2-4}}$$

$$= \lim_{x\to-\infty}\frac{x[(x-8)^2-(x^2-4)]}{\sqrt{x^2-4}\cdot[(x-8)-\sqrt{x^2+4}]} = \lim_{x\to-\infty}\frac{x(-16x+68)}{\sqrt{x^2-4}\cdot[(x-8)-\sqrt{x^2+4}]} = \frac{-16}{-2} = 8. \ \text{(1 pt)}$$

Therefore, the line $y = x-8$ and $y = -x+8$ are the slant asymptote of $f(x)$.

(b) Compute

$$f'(x) = \frac{(x+4)(x^2-4x+8)}{(x^2-4)^{\frac{3}{2}}}. \ \text{(1 pt)}$$

By first derivative test, $f(x)$ is increasing on $(-4,-2)\cup(2,\infty)$ (1 pt) and $f(x)$ is decreasing on $(-\infty,-4)$. (1 pt) The The local minimum point of $f(x)$ is $(-4, 8\sqrt{3})$ (1 pt) and no local maximum occurs. (1 pt)

(c) After solving $f''(x) = 0$, we have $x = 12 \pm 2\sqrt{34}$. Note that $0 < 12 - 2\sqrt{34} = \frac{8}{12+2\sqrt{34}} < 2$. So $f(x)$ is concave upward on $(-\infty, -2) \cup (12 + 2\sqrt{34}, \infty)$ (1 pt) and concave downward on $(2, 12 + 2\sqrt{34})$. (1 pt) The inflection point occurs at $x = 12 + 2\sqrt{34}$. (1 pt)

(d) The graph is as following: